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INTEGRAL INEQUALITIES FOR DIFFERENTIABLE HARMONICALLY (s, m)-PREINVEX FUNCTIONS

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ABSTRACT. In this paper, we define a new generalized class of preinvex functions which includes harmonically (s, m)-convex functions as a special case and establish a new identity. Using this identity, we introduce some new integral inequalities for harmonically (s, m)-preinvex functions.

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1. Introduction

In this section, we recall some basic concepts, properties and results in the convex analysis. For more details, see [1, 2] and the references therein. Let K be a set in the finite dimensional Euclidean space $\mathbb{R}^n$,whose inner product and norm are denoted by $\langle ., . \rangle$ and $\|.\|$ respectively.

Definition 1.1. A set K in $\mathbb{R}^n$ is said to be a convex set, if and only if,

$$(1-t)u + tv \in K, \quad for\ all\ u, v \in K\ ,\ t \in [0, 1].$$

Definition 1.2. A function f on the convex set K is said to be a convex function if and only if

$$f((1-t)u + tv) \leq (1-t)f(u) + tf(v), \quad for\ all\ u, v \in K\ ,\ t \in [0, 1].$$

For the differentiable convex function, we have the following interesting result.

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Theorem 1.3. [3] *Let K be a nonempty convex set in $\mathbb{R}^n$, and let f be a differentiable convex function on the set K. Then $u \in K$ is the minimum of f if and only if $u \in K$ satisfies the inequality*

$$\langle f'(u), v-u\rangle \geq 0, \ for\ all\ v \in K.$$

Definition 1.4. [4] A set $K_\eta \subseteq \mathbb{R}$ is said to be invex set with respect to the bifunction $\eta(.,.)$ if and only if

$$x + t\eta(y,x) \in K_\eta, \ for\ all\ x, y \in K_\eta, \ t \in [0,1].$$

The invex set K_η is also called η-connected set. Note that, if $\eta(b,a) = b-a$, then invex set becomes the convex set. Clearly, every convex set is an invex set but converse is not true in general.

Definition 1.5. [5] Let K_η be an invex set in $\mathbb{R}$. Then, a function $f : \mathbb{R} \to \mathbb{R}$ is said to be preinvex function with respect to the bifunction $\eta(.,.)$ if and only if

$$f(x + t\eta(y,x)) \leq (1-t)f(x) + tf(y) \ \ for\ all\ x, y \in K_\eta, \ t \in [0,1].$$

Theorem 1.6. [6] *Let K_η be an invex set in $\mathbb{R}$ and let f be a differentiable preinvex function on set K_η. Then $u \in K_\eta$ is the minimum of f if and only if $u \in K_\eta$ satisfies the inequality*

$$\langle f'(u), \eta(v,u)\rangle \geq 0, \ for\ all\ v \in K_\eta.$$

Definition 1.7. [7] A set $K_h \subset \mathbb{R}/\{0\} \to \mathbb{R}$ is said to be a harmonically convex set if and only if

$$\frac{uv}{v + t(u-v)} \in K_h, \ \ for\ all\ u, v \in K_h, \ t \in [0,1].$$

Definition 1.8. [7] A function $f : K_h \subset \mathbb{R}/\{0\} \to \mathbb{R}$ is said to be harmonically convex function if and only if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq (1-t)f(x) + tf(y), \ \ for\ all\ x, y \in K_h, \ t \in [0,1].$$

Definition 1.9. [8] The function $f : I \subset (0,\infty) \to \mathbb{R}$ is said to be harmonically (s,m)-convex in second sense, where $s \in (0,1]$ and $m \in (0,1]$ if

$$f\big(\frac{mxy}{mty + (1-t)x}\big) = f\big((\frac{t}{x} + \frac{1-t}{my})^{-1}\big) \leq t^s f(x) + m(1-t)^s f(y)$$

$\forall x, y \in I$ and $t \in [0,1]$.

Remark 1.1. Note that for $s = 1$, (s,m)-convexity reduces to harmonically m-convexity and for $m = 1$, harmonically (s,m)-convexity reduces to harmonically s-convexity in second sense and for $s, m = 1$, harmonically (s,m)-convexity reduces to ordinary harmonically convexity .

Definition 1.10. [9] A set $I = [a, a + \eta(b,a)] \subseteq \mathbb{R}/\{0\}$ is said to be a harmonic invex set with respect to the bifunction $\eta(,)$ if and only if

$$\frac{x(x + \eta(y,x))}{x + (1-t)\eta(y,x)} \in I, \ \ for\ all\ x, y \in I, \ t \in [0,1]$$

Definition 1.11. [1] Let $h : [0,1] \subseteq J \to \mathbb{R}$ be a non-negative function. A function $f : I \to [a, a+\eta(b,a)] \subseteq \mathbb{R}/\{0\} \to \mathbb{R}$ is relative harmonic preinvex function with respect to an arbitrary nonnegative function h and an arbitrary bifunction $\eta(,)$ if

$$f\left(\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)}\right) \leq h(1-t)f(x) + h(t)f(y), \quad for\ all\ x,y \in I,\ t \in [0,1]$$

2. Main Results

Now, we define the class of harmonically (s,m)-preinvex functions which is motivated by the definition of harmonically (s,m)-convex functions defined by I. A. Baloch et al. [8].

Definition 2.1. A function $f : [a, a+\eta(b,a)] \subseteq \mathbb{R}/\{0\} \to \mathbb{R}$ is said to be harmonically (s,m)-preinvex functions with respect to the bifunction $\eta(,)$, if

$$f\left(\frac{x(x+\eta(my,x))}{x+t\eta(my,x)}\right) = f\left(\frac{t}{x} + \frac{1-t}{x+\eta(my,x)}\right)^{-1} \leq t^s f(x) + m(1-t)^s f(y)$$

for all $x, y \in [a, a+\eta(b,a)]$, with $x < my$, $t \in [0,1]$, $s \in (0,1]$, $m \in (0,1]$.

Note that, if $\eta(y,x) = y-x$, then harmonic (s,m)-preinvexity reduce to harmonic (s,m)-convexity.
We need the following identity, which plays an important role in the derivations of our main results.

Lemma 2.2. *Let $f : [a, a+\eta(mb,a)] \subseteq \mathbb{R}/\{0\} \to \mathbb{R}$ be a differentiable function on the interior of I° of I. If $f' \in [a, a+\eta(mb,a)]$ and $\lambda \in [0,1]$, then*

$$\begin{aligned} &\Upsilon_f(a, a+\eta(mb,a);\lambda) \\ = \quad & \frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\left[\int_0^{\frac{1}{2}} \frac{\lambda-2t}{(a+t\eta(mb,a))^2} f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)dt\right. \\ &\left. + \int_{\frac{1}{2}}^1 \frac{2-2t-\lambda}{(a+t\eta(mb,a))^2} f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)dt\right] \end{aligned}$$

where

$$\begin{aligned} &\Upsilon_f(a, a+\eta(mb,a);\lambda) \\ &= (1-\lambda)f\left(\frac{2a(a+\eta(mb,a))}{2a+\eta(mb,a)}\right) + \lambda\left[\frac{f(a)+f(a+\eta(mb,a))}{2}\right] \\ &\quad - \frac{2a(a+\eta(mb,a))}{\eta(mb,a)} \int_a^{a+m\eta(b,a)} \frac{f(x)}{x^2}dx \end{aligned}$$

Proof. Integrating by parts, we have

$$I_1 = \frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\int_0^{\frac{1}{2}} \frac{\lambda-2t}{(a+t\eta(mb,a))^2} f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)dt$$

$$= \frac{1-\lambda}{2} f\left(\frac{2a(a+\eta(mb,a))}{2a+\eta(mb,a)}\right) + \frac{\lambda}{2} f(a+\eta(mb,a))$$
$$- \int_0^{\frac{1}{2}} f\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right) dt,$$

and

$$I_2 = \frac{a(a,a+\eta(mb,a))\eta(b,a)}{2} \int_{\frac{1}{2}}^1 \frac{2t-2+\lambda}{(a+t\eta(mb,a))^2} f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right) dt$$
$$= \frac{1-\lambda}{2} f\left(\frac{2a(a+\eta(mb,a))}{2a+\eta(mb,a)}\right) + \frac{\lambda}{2} f(a) - \int_{\frac{1}{2}}^1 f\left(\frac{a(a+\eta(mb,a))}{x+t\eta(mb,a)}\right) dt$$

Thus

$$I_1 + I_2$$
$$= (1-\lambda) f\left(\frac{2a(a+\eta(mb,a))}{2a+\eta(mb,a)}\right) + \lambda \left[\frac{f(a)+f(a+\eta(mb,a))}{2}\right]$$
$$- \frac{2a(a+\eta(mb,a))}{\eta(mb,a)} \int_a^{a+m\eta(b,a)} \frac{f(x)}{x^2} dx$$

which is the required result. □

Theorem 2.3. *Let $f : [a, a+\eta(mb,a)] \subseteq \mathbb{R}/\{0\} \to \mathbb{R}$ be a differentiable function on the interior I° of I. If $f' \in [a, a+\eta(mb,a)]$ and $|f'|^q$ is harmonic (s,m)-preinvex function on I for $q \geq 1$ and $\lambda \in [0,1]$, then*

$$|\Upsilon_f(a, a+\eta(mb,a); \lambda)|$$
$$\leq \frac{a(a+\eta(mb,a))\eta(mb,a)}{2} \sigma_1(a,b;\lambda)^{1-\frac{1}{q}} \{\sigma_2(a,b;\lambda,s)|f'(a)|^q$$
$$+ m\sigma_3(a,b;\lambda,s)|f'(b)|^q\}^{\frac{1}{q}}$$
$$+ \sigma_4(a,b;\lambda)^{1-\frac{1}{q}} \{\sigma_5(a,b;\lambda,s)|f'(a)|^q + m\sigma_6(a,b;\lambda,s)|f'(b)|^q\}^{\frac{1}{q}},$$

where one can evaluate these integrals using any mathematical software (i.e maple).

$$\sigma_1(a,b;\lambda) = \int_0^{\frac{1}{2}} \frac{|\lambda-2t|}{(a+t\eta(mb,a))^2} dt,$$

$$\sigma_2(a,b;\lambda,s) = \int_0^{\frac{1}{2}} \frac{|\lambda-2t|(1-t)^s}{(a+t\eta(mb,a))^2},$$

$$\sigma_3(a,b;\lambda,s) = \int_0^{\frac{1}{2}} \frac{|\lambda-2t|t^s}{(a+t\eta(mb,a))^2},$$

$$\sigma_4(a,b;\lambda) = \int_{\frac{1}{2}}^1 \frac{|2-2t-\lambda|}{(a+t\eta(mb,a))^2} dt$$

$$\sigma_5(a,b;\lambda,s) = \int_0^{\frac{1}{2}} \frac{|2-2t-\lambda|(1-t)^s}{(a+t\eta(mb,a))^2},$$

$$\sigma_6(a,b;\lambda,s)=\int_0^{\frac{1}{2}}\frac{|2-2t-\lambda|t^s}{(a+t\eta(mb,a))^2}.$$

Proof. Using Lemma 2.2 and the power mean inequality, we have

$$\begin{aligned}
&|\Upsilon_f(a,a+\eta(mb,a);\lambda)|\\
&\leq \frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\int_0^{\frac{1}{2}}\frac{|\lambda-2t|}{(a+t\eta(mb,a))^2}\left|f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)\right|dt\\
&\quad+\int_{\frac{1}{2}}^1\frac{|2-2t-\lambda|}{(a+t\eta(mb,a))^2}\left|f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)\right|dt\\
&\leq \frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\left(\int_0^{\frac{1}{2}}\frac{|\lambda-2t|}{(a+t\eta(mb,a))^2}dt\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_0^{\frac{1}{2}}\frac{|\lambda-2t|}{(a+t\eta(mb,a))^2}\left|f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)\right|^q dt\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{2}}^1\frac{|2-2t-\lambda|}{(a+t\eta(mb,a))^2}\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_{\frac{1}{2}}^1\frac{|2-2t-\lambda|}{(a+t\eta(mb,a))^2}\left|f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)\right|^q dt\right)^{\frac{1}{q}}\\
&\leq \frac{a(a+\eta(mb,a))\eta(b,a)}{2}\left(\int_0^{\frac{1}{2}}\frac{|\lambda-2t|}{(a+t\eta(mb,a))^2}dt\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_0^{\frac{1}{2}}\frac{|\lambda-2t|}{(a+t\eta(mb,a))^2}\{t^s|f'(a)|^q+m(1-t)^s|f'(b)|^q\}dt\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{2}}^1\frac{|2-2t-\lambda|}{(a+t\eta(mb,a))^2}\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_{\frac{1}{2}}^1\frac{|2-2t-\lambda|}{(a+t\eta(mb,a))^2}\{t^s|f'(a)|^q+m(1-t)^s|f'(b)|^q\}dt\right)^{\frac{1}{q}}\\
&=\frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\sigma_1(a,b;\lambda)^{1-\frac{1}{q}}\{\sigma_2(a,b;\lambda,s)|f'(a)|^q\\
&\quad+m\sigma_3(a,b;\lambda,s)|f'(b)|^q\}^{\frac{1}{q}}\\
&\quad+\sigma_4(a,b;\lambda)^{1-\frac{1}{q}}\{\sigma_5(a,b;\lambda,s)|f'(a)|^q+m\sigma_6(a,b;\lambda,s)|f'(b)|^q\}^{\frac{1}{q}},
\end{aligned}$$

which is the required result. □

If $q=1$, then Theorem 1.6 reduces to the following result, which appears to be a better new one than already exists.

Corollary 2.4. *Let $f : [a, a+\eta(mb,a)] \subseteq \mathbb{R}/\{0\} \to \mathbb{R}$ be a differentiable function on the interior of I° of I. If $f' \in [a, a+\eta(mb,a)]$ and $|f'|$ is harmonic (s,m)-preinvex function on I and $\lambda \in [0,1]$, then*

$$\begin{aligned}
&|\Upsilon_f(a, a+\eta(mb,a);\lambda)| \\
&\leq \frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\{\sigma_2(a,b;\lambda,s)+m\sigma_3(a,b;\lambda,s)\}|f'(a)| \\
&+\{\sigma_5(a,b;\lambda,s)+m\sigma_6(a,b;\lambda,s)\}|f'(b)|,
\end{aligned}$$

where $\sigma_2(a,b;\lambda,s)$, $\sigma_3(a,b;\lambda,s)$, $\sigma_5(a,b;\lambda,s)$, $\sigma_6(a,b;\lambda,s)$ are given as in Theorem 1.6.

Theorem 2.5. *Let $f : [a, a+\eta(mb,a)] \subseteq \mathbb{R}/\{0\} \to \mathbb{R}$ be a differentiable function on the interior of I° of I. If $f' \in [a, a+\eta(mb,a)]$ and $|f'|^q$ is harmonic (s,m)-preinvex function on I for $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\lambda \in [0,1]$, then*

$$\begin{aligned}
&|\Upsilon_f(a, a+\eta(mb,a);\lambda)| \\
&\leq \frac{a(a+\eta(mb,a))\eta(mb,a)}{2}(\sigma_7(a,b;\lambda,p))^{\frac{1}{p}} \\
&\times\left(\{(1-\frac{1}{2^{s+1}})|f'(a)|^q+\frac{m}{2^{s+1}}|f'(b)|^q\}\right)^{\frac{1}{q}} \\
&+(\sigma_8(a,b;\lambda,p))^{\frac{1}{p}}\left(\{m(1-\frac{1}{2^{s+1}})|f'(b)|^q+\frac{1}{2^{s+1}}|f'(a)|^q\}\right)^{\frac{1}{q}},
\end{aligned}$$

where

$$\sigma_7(a,b;\lambda,p)=\int_0^{\frac{1}{2}}\frac{|\lambda-2t|^p}{(a+t\eta(mb,a))^{2p}}dt,$$
$$\sigma_8(a,b;\lambda,p)=\int_{\frac{1}{2}}^1\frac{|2-2t-\lambda|^p}{(a+t\eta(mb,a))^{2p}}dt.$$

Proof. Using Lemma 2.2 and Holder's integral inequality, we have

$$\begin{aligned}
&|\Upsilon_f(a, a+\eta(mb,a);\lambda)| \\
&\leq \frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\int_0^{\frac{1}{2}}\frac{|\lambda-2t|}{(a+t\eta(mb,a))^2}\left|f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)\right|dt \\
&+\int_{\frac{1}{2}}^1\frac{|2-2t-\lambda|}{(a+t\eta(mb,a))^2}\left|f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)\right|dt \\
&\leq \frac{a(a,a+\eta(mb,a))\eta(mb,a)}{2}\left(\int_0^{\frac{1}{2}}\frac{|\lambda-2t|^p}{(a+t\eta(mb,a))^{2p}}dt\right)^{\frac{1}{p}} \\
&\times\left(\int_0^{\frac{1}{2}}\left|f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)\right|^q dt\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\int_{\frac{1}{2}}^{1} \frac{|2-2t-\lambda|^p}{(a+t\eta(mb,a))^{2p}}dt\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^{1} \left|f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{a(a+\eta(mb,a))\eta(b,a)}{2}\left(\int_0^{\frac{1}{2}} \frac{|\lambda-2t|^p}{(a+t\eta(mb,a))^{2p}}dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^{\frac{1}{2}} \{t^s|f'(a)|^q + m(1-t)^s|f'(b)|^q\}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1} \frac{|2-2t-\lambda|^p}{(a+t\eta(mb,a))^{2p}}dt\right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^{1} \{t^s|f'(a)|^q + m(1-t)^s|f'(b)|^q\}dt\right)^{\frac{1}{q}}$$

$$= \frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\left(\int_0^{\frac{1}{2}} \frac{|\lambda-2t|^p}{(a+t\eta(mb,a))^{2p}}dt\right)^{\frac{1}{p}}$$

$$\times\left(\{(1-\frac{1}{2^{s+1}})|f'(a)|^q + m\frac{1}{2^{s+1}}|f'(b)|^q\}\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1} \frac{|2-2t-\lambda|^p}{(a+t\eta(mb,a))^{2p}}dt\right)^{\frac{1}{p}}$$

$$\times\left(\{m(1-\frac{1}{2^{s+1}})|f'(b)|^q + \frac{1}{2^{s+1}}|f'(a)|^q\}\right)^{\frac{1}{q}}$$

The proof completes. □

Theorem 2.6. *Let $f : [a, a+\eta(mb, a)] \subseteq \mathbb{R}/\{0\} \to \mathbb{R}$ be a differentiable function on the interior I° of I. If $f' \in [a, a+\eta(mb, a)]$ and $|f'|^q$ is harmonic (s, m)-preinvex function on I for $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\lambda \in [0, 1]$, then*

$$|\Upsilon_f(a, a+\eta(mb, a); \lambda)|$$

$$\leq \frac{a(a+\eta(mb,a))\eta(mb,a)}{2} \times \left(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{2(p+1)}\right)^{\frac{1}{p}}$$

$$\times(\sigma_9(a,b;\lambda,q)|f'(a)|^q + m\sigma_{10}(a,b;\lambda,q)|f'(b)|^q)^{\frac{1}{q}}$$

$$+(\sigma_{11}(a,b;\lambda,q)|f'(a)|^q + m\sigma_{12}(a,b;\lambda,q)|f'(b)|^q)^{\frac{1}{q}},$$

where

$$\sigma_9(a,b;\lambda,p) = \int_0^{\frac{1}{2}} \frac{t^s}{(a+t\eta(mb,a))^{2q}}dt$$

$$\sigma_{10}(a,b;\lambda,p) = \int_0^{\frac{1}{2}} \frac{(1-t)^s}{(a+t\eta(mb,a))^{2q}}dt$$

$$\sigma_{11}(a,b;\lambda,p)=\int_{\frac{1}{2}}^{1}\frac{t^{s}}{(a+t\eta(mb,a))^{2q}}dt$$
$$\sigma_{12}(a,b;\lambda,p)=\int_{\frac{1}{2}}^{1}\frac{(1-t)^{s}}{(a+t\eta(mb,a))^{2q}}dt$$

Proof. Using Lemma 2.2 and the Holder's integral inequality, we have

$$\begin{aligned}
&|\Upsilon_f(a,a+\eta(mb,a);\lambda)|\\
&\le \frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\int_0^{\frac{1}{2}}|\lambda-2t|\left|\frac{f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)}{(a+t\eta(mb,a))^2}\right|dt\\
&\quad+\int_{\frac{1}{2}}^{1}|2-2t-\lambda|\left|\frac{f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)}{(a+t\eta(mb,a))^2}\right|dt\\
&\le \frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\left(\int_0^{\frac{1}{2}}|\lambda-2t|^p dt\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^{\frac{1}{2}}\left|\frac{f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)}{(a+t\eta(mb,a))^2}\right|^q dt\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{2}}^{1}|2-2t-\lambda|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left|\frac{f'\left(\frac{a(a+\eta(mb,a))}{a+t\eta(mb,a)}\right)}{(a+t)\eta(mb,a))^2}\right|^q dt\right)^{\frac{1}{q}}\\
&\le \frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\left(\int_0^{\frac{1}{2}}|\lambda-2t|^p dt\right)^{\frac{1}{p}}\\
&\times\left(|f'(a)|^q\int_0^{\frac{1}{2}}\frac{t^s}{(a+t\eta(mb,a))^{2q}}dt+m|f'(b)|^q\int_0^{\frac{1}{2}}\frac{(1-t)^s}{(a+t\eta(mb,a))^{2q}}dt\right)^{\frac{1}{q}}\\
&\quad+\left(\int_{\frac{1}{2}}^{1}|2-2t-\lambda|^p dt\right)^{\frac{1}{p}}\\
&\times\left(|f'(a)|^q\int_{\frac{1}{2}}^{1}\frac{t^s}{(a+t\eta(mb,a))^{2q}}dt+m|f'(b)|^q\int_{\frac{1}{2}}^{1}\frac{(1-t)^s}{(a+t\eta(mb,a))^{2q}}dt\right)^{\frac{1}{q}}\\
&=\frac{a(a+\eta(mb,a))\eta(mb,a)}{2}\times\left(\frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{2(p+1)}\right)^{\frac{1}{p}}\\
&\times(\sigma_9(a,b;\lambda,q)|f'(a)|^q+m\sigma_{10}(a,b;\lambda,q)|f'(b)|^q)^{\frac{1}{q}}\\
&+(\sigma_{11}(a,b;\lambda,q)|f'(a)|^q+m\sigma_{12}(a,b;\lambda,q)|f'(b)|^q)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. □

3. Conclusion

In this paper, we have studied the class of Harmonically (s, m)-preinvex functions which is generalization of Harmonically preinvex functions and have established similar results to Hermite-Hadamard inequalities for this class. Class of functions defined in this paper may stimulate further research in this field. The interested researchers are encouraged to find the particular examples of new class of functions presented in this paper.

Competing Interests

The author(s) do not have any competing interests in the manuscript.

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