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On Some Topological Properties of $C_h(X)$ and of the Dual Operator

Zéphirin N'teba Makala [a++], Muaku Mvunzi [a#], Rostin Mabela Makengo [b†], Alain Musesa Landa [b†], Gérard Tawaba Musian Tayen [a‡], Freddy Tsasa Bakweno [c^], Lady Nlandu Mabumbi [b##], Emilien LO Ranu Londjiringa [d##], Camile Likotelo Binene [a##*] and Cédric Kabeya Tshiseba [a#^]

[a] Department of Mathematics and Computer Science, Faculty of Science and Technology, National Pedagogical University, Kinshasa, Democratic Republic of the Congo.
[b] Department of Mathematics and Computer Science, Faculty of Science and Technology, University of Kinshasa, Kinshasa, Democratic Republic of the Congo.
[c] Department of Mathematics, Exact Sciences Section, ISP Popokabaka, Bandundu, Democratic Republic of the Congo.
[d] Department of Mathematics and Physics, Exact Sciences Section, ISP BUNIA, Ituri, Democratic Republic of the Congo.

Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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[++] Doctoral student;
[#] Faithful Professor;
[†] Ordinary Professor;
[‡] Professor;
[^] Work Manager;
[##] Assistant;
[#^] Associate Professor;
**Corresponding author: Email: camileliko@gmail.com;*

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Abstract

The hypo-topology on the algebra C(X) of real-valued continuous functions defined on a Tychonoff space 2^X $f \in C(X)$ with its hypograph, $hypof = \{(x,t) \in X \text{ x } \mathbb{R}: f(x) \geq t\}$. This topology is very useful in the calculus of variations and in optimization theory (e.g. Maximization problems). We denote C(X) with the hypo-topology by $C_h(X)$. Our study deals with fundamental properties of these function spaces, and with the linear operators on them and as well as the characterization of the topological properties of $C_h(X)$in terms of topological properties of the base space X. We are studying the linear operator between the functional algebras $C_h(X)$ $and\ the$ $C_h(Y)$. We are primarily concerned with the continuity of the evaluation functional, the general evaluation and the continuity of the characters of $C_h(X)$before the investigation of the properties of the dual operator $f^*$of a continuous function $f: X \to Y$. This operator is defined by $f^*: C_h(Y) \to C_h(X)$, where $f^*(g) = gof$ for all $g \in C(Y)$. The continuity of f^*enables us to characterize the continuity of an algebra homomorphism of the type $\varphi: C_h(Y) \to C_h(X)$ for a realcompact space Y. For such a space, we present a type of Riesz theorm with states that an algebra homomorphism $\varphi: C_h(Y) \to C_h(X)$is continuous if and only if there exists a unique hypo-function $f: X \to Y$ such that $\varphi = f^*$. There after, we give the equivalence between the properties of $f$and those of f^*. The study of the continuous linear functional on $C_h(X)$ helps us to compute the topological dual space of this algebra. We show here that this dual space is useful only when the set of isolated points of X is dense.

Keywords: Hypo-topology; hypograph; hypo-function; π-basis.

1 Introduction

For a given 2^XTychonoff space X, we denote by the set of all closed subsets of X. The Fell topology on $2^X$is defined by the open subbases of the form

$$U^- = \{A \in 2^X: A \cap U \neq \phi\}\ et\ V^+ = \{A \in 2^X: A \subseteq V\}$$

where U is an open set of X and $X \backslash V$is a compact set of X.

Let be $f: X \to \mathbb{R}$a function. We define the hypograph $f$by

$$hypof = \{(x,t) \in X \text{ x } \mathbb{R}: f(x) \geq t\}.$$

If in addition, $f$is semicontinuous above (respectively below) $hypof \in 2^{X \text{ x } \mathbb{R}}$. We define the Fell topology on C(X), called hypo-topology by identifying each $f \in C(X)$with its hypograph. We denote C(X) equipped with the hypo-topology by $C_h(X)$.

An element of the subbase of the open sets of $C_h(X)$ is of the form

$$U^- = \{f \in C(X): (hypof) \cap U \neq \phi\}\ et$$
$$V^+ = \{f \in C(X): (hypof \subseteq V\}$$

where U is an open subset of X x $\mathbb{R}$ et (X x $\mathbb{R}$)\V is a compact subset of X x $\mathbb{R}$.

For a subset A of X and a subset V of $\mathbb{R}$, we set $[A,V] = \{f \in C(X): f(A) \subseteq V\}$. With this notation, let U be an open subset of X, K a compact subset of X and ℓan element of $\mathbb{R}$. We define

$$[U,\ell]^- = \{f \in C(X): \exists x \in U \text{ with } f(x) > \ell\}\ and$$
$$[K,\ell]^+ = \{f \in C(X): f(x) < \ell \text{ for everything } x \in K\}.$$

In such a way, $[U,\ell]^- = U\{[\{x\},(\ell,+\infty)]: x \in U\} = \left(U \text{ x } (\ell,+\infty)\right)^-$ and

$[K,\ell]^+ = \big((X \text{ x } \mathbb{R})\backslash(\text{Kx }\{\ell\})\big)^+ = [K,(-\infty,\ell)]$. Thus, the sets $[U,\ell]^-$and $[K,\ell]^+$are open in the hypo-topology. Here the intervals $(\ell,+\infty)\ et\ (-\infty,\ell)$represent the set of real numbers strictly greater than $\ell$and strictly less than ℓrespectively.

2 Results

This paper is an investigation of linear operators between algebras $C_h(X) et\ C_h(Y)$. We are particularly interested in linear operators between these algebras. We study the continuity of some remarkable functions in relation to hypo-topology (Arkhangel'skii 1992).

1. Properties of the dual operator. (McCoy and Ntantu 1998, McCoy and Ntantu 1992, McCoy and Ntantu 1995; Ntantu 1990, Gillman and Jerison 1976)

a. Injection of the dual operator

A map $f: X \to Y$is almost surjective if and only if $f(x)$ is a dense subset of Y.

b. Theorem

Let be $f: X \to Y$a continuous function and $f^*: C_h(Y) \to C_h(X)$ its dual operator. Then $f^*$is injective if and only if $f$is almost surjective.

Evidence

Suppose f^*injective. To show that $f(x)$is dense in Y, suppose the opposite, that is, . $\overline{f(x)} \neq Y$Let . $y \in Y\backslash \overline{f(x)}$Since Y is completely regular, it exists $g \in C(Y)$such that $g(y) = 1\ et\ g\big(\overline{f(x)}\big) = \{0\}$. It follows that, $g\big(f(x)\big) = \{0\}$that is, $(gof)(x) = \{0\}$.This means that, $gof\ = 0$that is , $f^*(g) = O_X = f^*(0_Y)$.

From $f^*(g) = f^*(0_Y)$we obtain $g = 0_Y$because $f^*$is injective. $g = 0_Y$contradicts the construction of$g$ because $g(Y) = 1 \neq 0$. In conclusion we must have $\overline{f(x)} = Y$and $f$is almost surjective.

Conversely, let us assume f that is almost surjective. To see that $f^*$is injective, let $g,h \in C(Y)$such that $f^*(g) = f^*(h)$.Let us show that $g = h$. Let therefore $y \in f(X)$. There exists $x \in X$such that

$$y = f(x).$$

From where,$g(y) = g\big(f(x)\big) = (gof)(x) = f^*(g)(x) = f^*(h)(x) = (hof)(x)$
$= h\big(f(x)\big) = h(Y)$.

So $g = h$on $f(X)$, hence the equality $g = h$on X.

This being true for everything(g,h) in $C(Y)$x $C(Y)$, we conclude that $f^*$is injective.■

2. Surjection of the dual operator

A subset A of a Tychonoff space is c-immersed in X if and only if every continuous and bounded application $f: A \to \mathbb{R}$admits a continuous extension $F: X \to \mathbb{R}$(i.e. $f(a) = F(a)$for all $a \in A$).

With this notion, we obtain the following result:

3. Theorem 1

Let be $f: X \to Y$a continuous function and $f^*: C_h(Y) \to C_h(X)$ its dual operator.
Then $f^*$is surjective if and only if $f$is a homeomorphism from X to its image $f(X)\ et\ f(X)$ is C-imme rgé in Y.

Evidence

Suppose f^*surjective. We first show that $f$is injective. To do this, let $x\; et\; x'$two elements of X be such that $f(x) = f(x')$.

Let us suppose for the sake of absurdity that $x \neq x'$. Since X is completely regular, there exists $g \in C(X)$ such that $g(x) = 1\; et\; g(x') = 0$.

By surjectivity of f^*, there exists $h \in C(Y)$ such that $g = f^*(h)$. Now,$1 = g(x) = f^*(h)(x) = (hof)(x) = h\big(f(x)\big) = h\big(f(x')\big)$

$= (hof)(x) = f^*(h)(x') = g(x') = 0$ is a contradiction (because $1 \neq 0$). Hence $x = x'$and $f$is injective.
Next, to show that $f: X \to f(X)$ is a homeomorphism, let us show that $f^{-1}: f(x) \to f(X)$is continuous. Let $y_0 \in f(X)$. Let us show that $f^{-1}$is continuous at y_0.

To do this, let V be a neighborhood of $f^{-1}(y_0)$ in X. We must find a neighborhood W of y_0 in $f(X) tel\; que\; f^{-1}(W) \subset V$. As $y_0 \in f(X)$, there exists $x_0 \in X$ such that $y_0 = f(x_0)$. Also, $f^{-1}(y_0) = f^{-1}\big(f(x_0)\big) = x_0$ and therefore $x_0 \in V$. As V is a neighborhood of x_0 in X and X is completely regular, there exists $g \in C(X)$ such that$g(x_0) = 0$ and $g(X \backslash V) = \{1\}$. By the surjectivity of f^*, there exists $h \in C(Y)$ such that $g = f^*(h)$.

NOW, $0 = g(x_0) = f^*(h)(x_0) = (hof)(x_0) = h\big(f(x_0)\big) = h(y_0)$.

By posing $W = h^{-1}(0,1) \cap f(X)$, we have that W is a neighborhood of y_0 in $f(X)$. We want to show that $f^{-1}(W) \subset V$. By calculating $f^{-1}(W)$, we have:

$f^{-1}(W) = f^{-1}[f(X) \cap h^{-1}(0,1)]$
$= f^{-1}\big(f(x)\big) \cap f^{-1}\big(h^{-1}(0,1)\big)$
$= X \cap (hof)^{-1}[0,1]$
$= (hof)^{-1}[0,1]$
$= \big(f^*(h)\big)^{-1}[(0,1)]$

To see that $f^{-1}(W) \subset V$, either $x \in f^{-1}(W) = \big(f^*(h)\big)^{-1}[0,1]$. So$f^*(h)(x) \in (0,1)$

Let us assume by absurdity that $x \notin V$. Then $x \in X \backslash V$and so $g(x) = 1$by the construction of g.

then turns out that $1 = g(x) f^*(h)(x) \in (0,1)$ is a contradiction because $1 \;\notin (0,1)$. Thus $x \in V$. This being true for all $x \in f^{-1}(W)$, we conclude that $f^{-1}(W) \subset V$and $f^{-1}$is continuous in $y_0$as desired.

But then $y_0$being arbitrary in $f(X), f^{-1}$is continuous on $f(X)$. Therefore $f$is a homeomorphism from X to$f(X)$.
Finally, it remains to prove that $f(x)$is C-immersed in Y.

Let be $g: f(x) \to \mathbb{R}$a bounded and continuous function. We must construct a continuous function $g^*: Y \to \mathbb{R}$ such that$\hat{g} = g$ on $f(x)$.

We have the composite: $X \to f(x) \to \mathbb{R}$, that is to say $gof: X \to \mathbb{R}$which is continuous being the composite of 2 continuous functions.

$f \qquad g$

So $gof \in C(X)$. By the surjectivity of f^*, there exists $g^* \in C(Y)$such that $f^*(g^*) = gof$, that is $g^* of = gof$. To see that $g^*$is the desired extension, let $y \in f(x)$. Then there exists $x \in X\; tel\; que\; y = f(x)$.

Hence $g(y) = g\big(f(x)\big) = (gof)(x) = (g^* of)(x) = g^*(f(x) = g^*(y)$, for everything $y \in f(x)$.

This means that $g = g^*$sur $f(X) et\; f(X)$is C-immersed in Y.

Conversely, suppose that $f: X \to f(X)$is a homeomorphism such that $f(X)$is C-immersed in Y. Let us show that $f^*$is surjective.
Let $g \in C(X)$.us consider the composite: $f(X) \xrightarrow{f^{-1}} X \to \mathbb{R}$clearly,

$gof^{-1} \in C\big(f(x)\big)$. Since $f(x)$C is immersed in Y, there exists

$h \in C(Y)$ such as $h = gof^{-1}$ on $f(X)$. We show that $g = \hat{f}(h)$. Indeed, if $x \in X$, then $f(x) \in f(X)$. Whence$h\big(f(x)\big) = (gof^{-1})\big(f(x)\big)$

$= g\big[f^{-1}\big(f(x)\big)\big] = g(x)$, that's to say$g(x) = h\big(f(x)\big) = (hof)(x)$
$= f^*(h)(x)$for everything $x \in X$.

So, we have $g = f^*(h)$. This being true for everything $g \in C(X)$, $f^*$is surjective. ■

Theorem 2

f^* is bijective if and only if f is a homeomorphism from X onto $f(X)$ and $f(X)$ is dense and C-immersed in Y.

4. The almost surjection of the dual operator. (McCoy and Ntantu 1998, Dobrowolski et al. 1991)

A function $f: X \to Y$is a hypofunction if and only if for any open set V and any compact set K of X we have $f(K) \subset f(U) \Rightarrow K \subset U$.

Any injective application is a hypo-function.

Theorem

Let be $f: X \to Y$a function and $f^*: C_h(Y) \to C_h(X)$ its dual operator. Then $f^*$is almost surjective if and only if $f$is a hypo-function.

Evidence

Suppose that $f^*$is a hypofunction. Let us show that $f^*\big(C_h(Y)\big)$is dense in $C_h(X)$. To do this, it suffices to establish that any open set with a non-empty basis in$C_h(X)$ intersects $f^*\big(C_h(Y)\big)$. So let

$B = [U_1, \mathscr{s}_1]^- \cap \ldots \cap [U_m, \mathscr{s}_m]^- \cap [K_1, t_1]^+ \cap \ldots \cap [K_m, t_m]^+$an open set with a nonempty base in $C_h(X)$.
Either$I = \{1, \ldots, m\} et\ J = \{1, \ldots, n\}$. Let $i \in I$. Let us define$J_i, p_i, q_i, K_i, x_i et\ g_i$

in the following manner. First either$J_i = \{j \in J : t_j \leq \mathscr{s}_i\}$.

$Choisissons\ p_i, q_i$, in $\mathbb{R}$such that $\mathscr{s}_i < p_i et\ q_i < p_i$, and also such that $p_i < \min\{t_j : j \in J \setminus J_i\}$if $J \setminus J_i \neq \phi\ et\ q_i < \min\{t_j : j \in J_i\}\ si\ J_j \neq \phi$.
Let's ask$K_i' = \begin{cases} \cup\{K_j : j \in J_i\} si\ J_i \neq \phi \\ \phi \qquad si\ J_i \neq \phi \end{cases}$

Since B is not empty, then $U_i \not\subset K_i'$.Since f is a hypo-function, there exists $x_i \in U_i \backslash K_i$such that $f(x_i) \notin f(K_i')$. Finally, let be $g_i: Y \to [q_i, p_i]$a continuous function such that $g_i\big(f(x_i)\big) = q_i\ et\ g_i(Y) = p_i$for all $y \in f(K_i')$. We then define $g: Y \to \mathbb{R}$by $g(y) = \max\{g_i(y) : i \in I\}$ for all $y \in Y$. Clearly $g \in C_h(Y)$. We will then show that$f^*(g) \in B \cap f^*\big(C_h(Y)\big)$.

Let us show that $f^*(g) \in B$. For all $i \in I$, we know that $(gof)(x_i) = g\big(f(x_i)\big) \leq g_i\big(f(x_i)\big) = q_i > \mathscr{s}_i$.Hence $gof \in [U_i, \mathscr{s}_i]^-$for all $i \in I$.

Now let's take$i \in I, j \in J \; et \; x \in K_j$. If $j \in J_i$, then $x \in K_i'$; hence $g_i\big(f(x)\big) \in g_i\big(f(K_i')\big)$and thus $g_i\big(f(x)\big) = p_i < t_j$; such that $g_i\big(f(x)\big) < q_i < t_j$; which means that $gof \in \big[K_j, t_j\big]^+$ for all $j \in J$.Hence $f^*(g) = gof \in B$. So $B \cap f^*\big(C_h(Y)\big) \neq \phi$and so $f^*\big(C_h(Y)\big)$is dense in $C_h(Y)$.

Conversely, suppose that $f$is not a hypofunction. We will show that $f^*\big(C_h(Y)\big)$is not dense in $C_h(Y)$. Since $f$is not a hypofunction, there exists an open set U of X and a compact subset K of X such that$U \not\subset K \; mais$

$\phi(U) \subset \phi(K)$. Let us define $B = [U, 0]^- \cap [K, 0]^+$which is an open set of $C_h(X)$. Let $x_0 \in U \backslash K$, and be $g: x \to [-1,1]$a continuous function such that $g(x_0) = -1$and $g(x) = 1$for all $x \in K$.Then $g \in B$, and thus $B \neq \phi$.

Let us show that $f^*(K) \in B$for all $K \in C(Y)$that is to say that $K^*\big(C_h(Y)\big) \cap B = \phi$.Let us suppose for absurdity that this is not true . Let then be $k \in C_h(Y)$such that$kof \in B$.

As $kof \in [U, 0]^-$, there exists $x_1 \in U$such that $k\big(f(x_1)\big) > 0$.As $f(x_1) \in f(U) \subseteq f(k)$, there exists such $x_2 \in K$that $f(x_1) = f(x_2)$. Hence $k\big(f(x_2)\big) < 0$because $kof \in [K, 0]^+$; which contradicts the inequality

$$k\big(f(x)\big) > 0.$$

So $K^*\big(C_h(Y)\big) \cap B = \phi$and so $K^*\ \big(C_h(Y)\big)$is not dense in $C_h(X)$. ■

5. Embedding the dual operator. (McCoy and Ntantu 1998; McCoy and Ntantu 1995, God 1972, Hirsch and Lacombe 2009, Arkhangel'skii and Ponomarev 1984)

A continuous application $f: X \to Y$is a k-function if and only if every compact set of Y is an image of fa compact set of X. (That is, $\forall K$compact set of Y, there exists C compact set of X such that $K = f(C)$).

Theorem

Let be $f: X \to Y$a continuous function and $f^*: C_h(Y) \to C_h(X)$its dual operator. Then $f^*$is an embedding of $C_h(Y)$in $C_h(X)$if and only if $f$is a weakly open k-function.

Evidence

Let $\mathcal{R} = f^*\big(C(Y)\big)$. First suppose that $f^*: C_h(Y) \to \mathcal{R}$is a homeomorphism. By the continuity of f^*, $f$is already weakly open. It remains to show that $f$is a k-function. To do this, let A be a compact of X. Since $W = [A, (-\infty, 1)]$is an open neighborhood of 0_Y, then $f^*(W)$is an open neighborhood of 0_Xin $\mathcal{R}$. There exist compacts $K_1, K_2, \ldots, K_n$in X and nonempty open sets $U_1, U_2, \ldots, U_n$in X and reals $t_1, t_2, \ldots, t_n, S_1, S_2, \ldots, S_n$such that

$$0_X \in [K_i, t_i]^+ \cap \ldots \cap [K_n, t_n]^+ \cap [U_i, S_i] \cap \ldots \cap \ [U_m, S_m] \cap \mathcal{R} \subset f^*(w).$$

For each $1 \leq i \leq n$, be $x_i \in U_i$.

assume $k = \{x_1, x_2, \ldots, x_n\} \cup K_1 \cup, \ldots, \cup K_m$ that is a compact of X. We first show that $A \subset f(K)$.Indeed, assuming the opposite, there would exist $a \in A \backslash f(K)$. Since Y is a Tychonoff space, there would exist $g: Y \to [0,1]$a continuous such that

$g(a) = 1 \; et \; g\big(f(K)\big) = \{0\}$.So $f^*(g) \in f^*(W)$ and so it would exist

$h \in W = [A, (-\infty, 1)]$such that $f^*(g) = f^*(h)$. By the injectivity of f^*, we will have $g = h$. From which $g(a) = h(a) < 1$on the one hand and, on the other hand by $g(a) = 1$the construction of g. With this contradiction, we must have $A \subset f(K)$. Now, by setting $C = K \cap f^{-1}(A)$, we have a compact of X such that $A = f(C)$. Therefore $f$is a k-function.

For the converse, suppose that $f: X \to Y$ is a weakly open k-function. We already know that $f^*: C_h(Y) \to C_h(X)$ is continuous by the previous theorem. Also, since f is surjective, f^* is injective. By setting $\mathcal{R} = f^*(C(Y))$, we have $f^*: C_h(Y) \to \mathcal{R}$ bijective and continuous. To have the desired homeomorphism, we show that $(f^*)^{-1}: \mathcal{R} \to C_h(Y)$ is continuous. For this, let $W_1 = [A_1, (-\infty, t)]$ an open set of the subbase of $C_h(Y)$, where A is a compact set of Y and $t \in \mathbb{R}$. Let us show that $((f^*)^{-1})^{-1}(W_1)$ is an open set of $\mathcal{R}$. But $((f^*)^{-1})^{-1}(W_1) = f^*(W_1)$.

Let $g \in f^*(W_1)$. Then there exists $h \in W_1$ such that $g = f^*(h) = hof$. Since f is a k-function and A_1 is a compact of Y, there exists C compact of X such that $A_1 = f(C)$. Then $g \in \mathcal{R} \cap [C, (-\infty, t)] \subset f^*(W_1)$ show that $f^*(W_1)$is a neighborhood of$g$ in $\mathcal{R}$. From where $f^*(W_1)$ is opened from $\mathcal{R}$.

Similarly, let be $W_2 = [U_2, S]$another open number of the subbase of $C_h(Y)$where $U_2$is a non-empty open number of Y and S a real number.

Either$g \in ((f^*)^{-1})^{-1}(W_2) = f^*(W_2) = f^*(U[Y, (S, +\infty)] : y \in U_2)$

$= U\, f^*([Y, (S, +\infty)])$. There exists $y_0 \in U_2$such that $g \in f^*([y_0, (S, +\infty)])$.Let $h \in [y_0, (S, +\infty)]$such that $g = f^*(h) = hof$. Also, as $f$is surjective and $y_0 \in U_2 \subset Y$, there exists $x_0 \in X$such that $f(x_0) = y_0 \in U_2$. Then $x_0 \in f^{-1}(U_2)$. Let us set $W_3 = [f^{-1}(U_2), 1]$. Then as

$g(x_0) = h(f(x_0)) = h(Y_0) > S$. SO$g \in [x_0, (S, +\infty)] \cap \mathcal{R} \subset [f^{-1}(U_2), S]$

$\cap\, \mathcal{R} = W_3 \cap \mathcal{R} \subset f^*(W_2)$. For the last inclusion, if $\ell \in W_3 \cap \mathcal{R}$, there exists $K \in C(Y)$such that $\ell = f^*(h) = hof$. $\ell \in W_3 = [f^{-1}(U_2), S]$shows that there exists $z \in f^{-1}(U_2)$such that $\ell(z) > S$. Now$f(z) \in U_2\ et$
$k = (f(z)) > S$implies that $k \in [U_2, S] = W_2$.

So, $f^*(k) \in f^*(W_2)$that is to say that $\ell \in f^*(W)$. We obtain the implication $W_3 \cap \mathcal{R} \subset f^*(W_2)$. From $g \in f^*(W_2)$, we draw

$g \in W_3 \cap \mathcal{R} \subset f^*(W_3)$; which means that $f^*(W_2)$is a neighborhood of $g$in $\mathcal{R}$for all $g \in f^*(W_2)$. From which $f^*(W_2)$is open of $\mathcal{R}$.

In conclusion, $(f^*)^{-1}: \mathcal{R} \to C_h(Y)$is continuous and thus $f^*: C_h(Y) \to C_h(X)$is an extension of $C_h(Y)$in$C_h(X)$. $\blacksquare$

It is known (see Dobrowolski et al. 1991) that $f^*: C_k(Y) \to C_k(X)$ is an embedding if and only if f is a k −function.

6. Homomorphism of algebras. (Gilsinger and Mohammed 2010; McCoy and Ntantu 1998; Auliac and Caby 2005; Dobrowolski and Mogilski 1992, Dobrowolski et al. 1990)

3 Continuity of homomorphisms of algebras

1. Definition

a) An application $\lambda : C(Y) \to C(X)$is a homomorphism of $\mathbb{R}$ −algebras if and only if$\lambda(1_Y) = 1_X, \lambda(\alpha f + \beta g) = \alpha\lambda(f) + \beta\lambda(g)\ et$

$\lambda\,(fg) = \lambda(f)\lambda(g)$for everything $f, g \in C(Y) et\ \alpha, \beta \in \mathbb{R}$.

Here $1_Y\ et\ 1_X$ are the unit elements of the algebras $C(Y) et\ C(X)$respectively.

Tychonoff space X in which every character of C(X) is of the form $e_x$where $x \in X$is called a full or real compact space. Among the full spaces, we can cite the compact spaces.

We equip C(Y) and C(X) with the hypo-topology and we obtain the following theorem:

2) Theorem

Let Y be a full space. An algebra homomorphism $\lambda: C_h(Y) \to C_h(X)$is continuous if and only if there exists a unique weakly open continuous function $f: X \to Y$such that $\lambda = f^*$.

Evidence

If $f$is continuous, weakly open such that $\lambda = f^*$, by Theorem 2, $\lambda = f^*$is continuous.

For the converse, let us assume $\lambda: C_h(Y) \to C_h(X)$continuous homomorphism of algebras. Let $x \in X$. Since $e_x o\lambda: C_h(Y) \to \mathbb{R}$is a character of $C_h(Y)$and Y is a full space, there exists $Y_X \in Y$such that $e_x o\lambda = e_{Y_X}$. We then define $f: X - Y$by $f(x) = Y_X$for all $x \in X$.

Now, let $g \in C(Y)$. Then $\lambda(g) \in C(X)$. From where for all $x \in X$, we have: $e_X(\lambda(g)) = (e_X o\lambda)(g) = e_{Y_X}(g) = e_{f(x)}(g) = g(f(x)) = (gof)(x)$.

So $\lambda(g)(x) = e_X(\lambda(g)) = (gof\ )(x)$that is to say that $\lambda(g) = gof = f^*(g)$for all $g \in C(Y)$. This shows that $\lambda = f^*$.

Moreover, since Y is completely regular and $gof = \lambda(g) \in C(X)$for all $g \in C(Y)$, then $f$becomes continuous.
Finally, the uniqueness of $f$follows from the fact that $C(Y)$separates the points of Y. Also, by the continuity of $\lambda = f^*$, $f$is weakly open. ■

3. Topological Dual OF$C_h(X)$. (Cauty et al. 1993; Fell 1992; El-Fattah eta l. 2002; McCoy and Ntantu 1998; Arenas 19999; Beer and Kenderov 1989)

3.1 The weak hypo-topology of$C_h(X)$

Although $C_h(X)$is not in general a topological vector space, we can however speak of its topological dual $C'_h(X)$by considering $C'_h(X) = \{\lambda: C_h(X) \to \mathbb{R}/\lambda$ is linear and continuous$\}$. The smallest (in the sense of inclusion) topology on C(X) that makes every element of continuous $C'_h(X)$is denoted by $C_S(X)$and is called the weak hypo-topology. This topology is useful only for spaces X having dense isolated points.

We now take topological spaces X whose set of isolated points $I_x$is dense in X. For $x \in X$, we consider the multiplicative linear form $e_x: C(X) \to \mathbb{R}$defined by $e_X(f) = f(x)$for all $f \in C(X)$. Then the weak hypo-topology is generated by thee_X Or $x \in I_X$.

An element of the subbase of open sets is of the form $[A, V]$where A is a finite subset of $I_X$and V is an open set of $\mathbb{R}$. It is clear that $C_S(X)$is the topology of simple convergence on isolated points of X.

We can also consider $C_S(X)$as a locally convex vector space whose topology is generated by the semi-norm $p_A: C(X) \to \mathbb{R}$defined by $p_A(f) = \sup\{|f(x)|: x \in A\}$where A is a finite subset of I_X. A basic neighborhood of $f$in $C_S(X)$is of the form $\langle f, A, \varepsilon \rangle = \{g \in C(X): |g(x) - f(x)| < \varepsilon, \forall x \in A\}$ where $A \subset I_X$, A is finite and $\varepsilon > 0$is a real number.

3.2 Topological dual of$C_h(X)$

The weak hypo-topology is both less fine than hypo-topology and the topology of simple convergence on X. To compute the topological dual $C'_S(X)\ de\ C_S(X)$, it is clear that $C'_h(X) \subset C'_S(X)$. Our goal is to show that there is equality between these two dual spaces.

1) Theorem

Let be $\lambda: C_S(X) \to \mathbb{R}$a non-zero and continuous linear form. Then there exists $x_1, x_2, \ldots, x_n$in $I_x$such that $\lambda$is a linear combination of e_{x_i}.

Evidence

As the open interval $(-1,1)$is a neighborhood of$\lambda(O_x) = 0$ in $\mathbb{R}$, by the continuity of λat the point O_x, there exists a finite subset $A = \{x_1, x_2, \dots, x_n\}$in $I_X$and $\varepsilon > 0$real such that $\lambda(\langle O_X, A, \varepsilon\rangle) \subset (-1,1)$.

We consider the $(n+1)$- linear forms $\lambda, e_{x_1}, e_{x_2}, \dots, e_{x_n}$on C(X). By a result of linear algebra either $\lambda$is a linear combination of e_{x_1}or then there exists $g \in C(X)$such that $\lambda(g) = 1\ et\ g \in \bigcap_{i=1}^{n+1} Ker\ e_{x_i}$. If there exists the same $g$then $g \in (\langle O_x, A, \varepsilon\rangle)$which would imply$1 = \lambda(g) \in \lambda(\langle O_x, A, \varepsilon\rangle) \subset (-1,1)$

With this contradiction, we conclude that λmust be a linear combination of e_{x_i}.

We have just shown that any continuous linear form on $C_S(X)$is also continuous on $C_h(X)$. From which we have the following corollary: ■

2) Corollary

$$C'_S(X) = C'_h(X) = \left\{\sum_{i=1}^{n} \alpha_i e_{x_i} : n \in \mathbb{N}, \alpha_i \in \mathbb{R}\ et\ x_i \in I_x\right\} = Eng\left(e_{I_x}\right)$$

Or $e_{I_x} = \{e_x : x \in I_x\}$.

4 Conclusion

We have now reached the end of our article, the aim of which was to study some topological properties of $C_h(X)$and of the dual operator as a function of the topological properties of the Tychonoff space X.

At the operator level, we exploited the continuity of some special functions, before tackling the dual operator which allowed us to characterize the continuity of a homomorphism of algebras of type $\varphi: C_h(Y) - C_h(X)$for a replete space Y. For such a space, $\varphi: C_h(Y) \to C_h(X)$is continuous if and only if there exists a unique hypo-function $f: X - Y$such that. For a $\varphi = f^*$continuous $f: X - Y$function, we define its dual application $f: C_h(Y) \to C_h(X)$by $f^*(g) = gof$, for all $g$in C(Y).

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text-to-image generators have been used during writing or editing of this manuscript.

Competing Interests

Authors have declared that no competing interests exist.

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