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Fermat and Pythagoras Divisors for a New Explicit Proof of Fermat's Theorem: $a^4 + b^4 = c^4$. Part I

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Abstract

In this paper we prove in a new way, the well known result, that Fermat's equation $a^4 + b^4 = c^4$, is not solvable in $\mathbb{N}$, when $abc \neq 0$. To show this result, it suffices to prove that: $(F_0): a_1^4 + (2^s b_1)^4 = c_1^4$, is not solvable in $\mathbb{N}$, (where $a_1, b_1, c_1 \in 2\mathbb{N}+1$, pairwise primes, with necessary $2 \le s \in \mathbb{N}$). The key idea of our proof is to show that if (F_0) holds, then there exist $\alpha_2, \beta_2, \gamma_2 \in 2\mathbb{N}+1$, such that $(F_1): \alpha_2^4 + (2^{s-1}\beta_2)^4 = \gamma_2^4$, holds too. From where, one conclude that it is not possible, because if we choose the quantity $2 \le s$, as minimal in value among all the solutions of (F_0), then $(\alpha_2, 2^{s-1}\beta_2, \gamma_2)$ is also a solution of Fermat's type, but with $2 \le s-1 < s$, witch is absurd. To reach such a result, we suppose first that (F_0) is solvable in $(a_1, 2^s b_1, c_1)$, $s \ge 2$ like above; afterwards, proceeding with "Pythagorician divisors", we creat the notions of "Fermat's *b*-absolute divisors": (d_b, d'_b) which it uses hereafter. Then to conclude our proof, we establish the following main theorem: there is an equivalence between (i) and (ii): (i) (F_0): $a_1^4 + (2^s b_1)^4 = c_1^4$, is solvable in $\mathbb{N}$, with $2 \le s \in \mathbb{N}$, $(a_1, b_1, c_1) \in (2\mathbb{N}+1)^3$, coprime in pairs. (ii) $\exists (a_1, b_1, c_1) \in (2\mathbb{N}+1)^3$, coprime in pairs, for wich: $\exists (b'_2, b_2, b''_2) \in (2\mathbb{N}+1)^3$ coprime in pairs, and $2 \le s \in \mathbb{N}$, checking $b_1 = b'_2 b_2 b''_2$, and such that for notations: $S = s - \lambda(s-1)$, with $\lambda \in \{0,1\}$ defined by $\frac{c_1 - a_1}{2} \equiv \lambda \pmod 2$,

$d_b = gcd\left(2^s b_1, c_1 - a_1\right) = 2^S b_2$ and $d'_b = 2^{s-S} b'_2 = \dfrac{2^s B_2}{d_b}$, where

$\left(2^s B_2\right)^2 = gcd\left(b_1^2, c_1^2 - a_1^2\right)$, the following system is checked:

$$\begin{cases} c_1 - a_1 = \dfrac{d_b^4}{2^{2+\lambda}} = 2^{2-\lambda}\left(2^{S-1} b_2\right)^4 \\ c_1 + a_1 = 2^{1+\lambda} d_b'^4 = 2^{1+\lambda}\left(2^{s-S} b'_2\right)^4 \\ c_1^2 + a_1^2 = 2 b_2''^4 \end{cases}$$; and this system implies:

$\left(b_{1-\lambda,2}^4\right)^2 + \left(2^{4s-3} b_{\lambda,2}^4\right)^2 = \left(b_2''^2\right)^2$; where: $\left(b_{1-\lambda,2}, b_{\lambda,2}, b''_2\right) = \begin{cases} \left(b'_2, b_2, b''_2\right) & \text{if } \lambda = 0 \\ \left(b_2, b'_2, b''_2\right) & \text{if } \lambda = 1 \end{cases}$;

From where, it is quite easy to conclude, following the method explained above, and which thus closes, part I, of this article.

Keywords

Factorisation in $\mathbb{Z}$, Greatest Common Divisor, Pythagoras Equation, Pythagorician Triplets, Fermat's Equations, Pythagorician Divisors, Fermat's Divisors, Diophantine Equations of Degree 2, 4-Integral Closure of $\mathbb{Z}$ in $\mathbb{Q}$

1. Introduction

1.1. Some Historical Reminders about Fermat's Theorem

In this paper, we are interested in some works of the french mathematician Pierre de Fermat (1601 - 1665), specially his well-known "solution" or problem of solving the equation: $a^n + b^n = c^n$, problem written by himself in the margin of his edition of Diophantus cf. [1], some day between 1621 and 1665, and called his Last Theorem, and who became world famous after his death.

From a historical point of view, to solve this problem, Fermat took advantage of, and promoted worldwide the so-called "infinite descent" method, which already existed in [1], that is a posthumous publication in 1670, 5 years after his death, by his son Samuel de Fermat. This method was originally used by Fermat himself, when solving the 20th Diophantus problem cf [1], i.e. "In $\mathbb{N}$: $uv\left(u^2 - v^2\right)$ can't be a square with $uv\left(u^2 - v^2\right) \neq 0$" i.e. to show the impossibility of squaring the area of entire right-angled triangles, and which is an effective means of proof. The usual and easy way to solve this problem is the one proposed in [2], which consist to proove that the equation $X^4 + Y^4 = Z^2$ obtained from $a^4 + b^4 = c^4$, is not solvable. The method used, is infinite descent . Following Fermat, many equally famous authors, ranging from Euler 1738 to Carmichael 1913, via Vranceanu 1966 (cf. [3] p. 15), proposed demonstrations all based on finality on this same method of infinite descent (except for certain demonstrations using the ring of integers of Gauss).

But in the end, the proof done in 1994 for the general case $a^p + b^p = c^p$, by the British mathematician Andrew Wiles cf. [4], somehow closed the problem of resolution.

This proof is due to the culmination of new methods developed in the 20th century, located at the common frontiers of algebraic number theory, arithmetic, algebraic geometry and complex analysis, focused on the properties of certain types of analytic functions called modular forms, dealing with certain conjectures about elliptic curves and modular forms (these conjectures were all fully proven, in 1986, 1994 and 1999).

It was in 1955 that Tanyama and Shimura announced their conjecture (rediscovered by Weil in 1967): which says that "Any elliptic curve is modular". In 1984, Hellegouarch and Frey, notice that it was possible to associate an elliptic curve with an eventual solution of Fermat, in this way: $y^2 = x\left(x - a^p\right)\left(x + \left(2^s b_1\right)^p\right)$, where $a^p + \left(2^s b_1\right)^p = c^p$; $a, b_1, c \in 2\mathbb{Z}+1$ with $a \equiv 3 \pmod 4$. This Hellegouarch-Frey elliptic curve with strange discriminant $\Delta = -\left(4a^p b^p c^p\right)^2$, is "semi-stable", and does not seem to be modular; indeed, this last hypothesis was proven by Ribet in 1986 cf. [5], using his proof for some cases of Serre's ε-conjecture; And it was finally in 1994 that the Tanyama-Shimura-Weil conjecture was proved by Wiles in 1994 cf. [4] [6], (for cases of semi-stable elliptic curves) which meant that Fermat's last theorem was totally proven.

1.2. The Case of Fermat's Theorem, for *n* = 4

In this article, we particularly focus on the case n=4, that is to say, in the resolution of the equation $a^4 + b^4 = c^4$, which as we know, only admits the solutions generated by the trivial solution $(1,0,1)$, and its associates (cf. Proposition 2.1.).

The question that still arises for solving this equation is the following:

Is there a Diophantine demonstration other than that: complex one? Wiles' type? or of the classic infinite descent using equation $X^4 + Y^4 = Z^2$? (This last one being closely linked to the resolution of many famous Diophantine equations of order 4, as for example, those related to the old problem of the impossibility of squaring the area of a right angled triangle).

The answer somewhere is yes, like for example in [7], where one can find infinite descent using only equations like $\alpha^4 + \beta^4 = \gamma^4$. But the calculations remain difficult, and no method is extracted from [7], which could serve as a tool to progress on other cases of Fermat's equations.

Conversely, one of the main objectives of our article is precisely to propose a standard resolution method, usable for $n = 4$, but also for $n = 2p$ or p, where p is prime. This method is based on some very particular divisors of a and b.

2. Utilities for the Proof of Fermat's Theorem for *n* = 4

2.1. Key Idea of the Proof for the Resolution of $a^4 + b^4 = c^4$

The goal of our article is to prove Theorem 3.1., i.e. Fermat's Theorem when $n = 4$, in the way that we are looking for a practical proof, using arithmetic invariants, that we will define, and known as valuable arithmetic tools, to have been used in our other works.

So we consider in $\mathbb{N}^*$, an eventual solution:

$$a_1^4 + \left(2^s b_1\right)^4 = c_1^4\text{, pairwise prime, } a_1 b_1 c_1 \text{ odd, and } s \geq 2\text{, being minimal in value.} \quad (2.1)$$

From there, as a consequence of the Pythagorician divisors Theorem, cf. Theorem 2.1. & Lemma 2.1., see also [8], and through our main result on Fermat's *b*-absolute divisors, and expressed in Theorem 2.2., it was possible for us to obtain a contradictory result, which is: There is another solution $\left(\alpha_1, \beta = 2^{s-1}\beta_1, \gamma_1\right)$, pairwise prime, $\alpha_1\beta_1\gamma_1$ odd, such that: $\alpha_1^4 + \left(2^{s-1}\beta_1\right)^4 = \gamma_1^4$; But this is impossible, because of the minimality condition of *s*, defined in (2.1).

Furthermore, we note that: In the case $s = 2$, we have: $\alpha_1^4 + \left(2\beta_1\right)^4 = \gamma_1^4$, which necessarily implies that $2\beta_1 \equiv 0 \pmod 4$, cf. Proposition 2.3.(i), which is impossible.

Roadmap and Articulations of the Proof

How to prove Theorem 3.1.? Consider a solution $\left(a_1, 2^s b_1, c_1\right)$ of equation (2.1.), and take a look at the even term $2^s b_1$, then necessarily a_1 and c_1 are odd, and therefore we can define $\lambda \in \{0,1\}$ such that $\frac{c-a}{2} \equiv \lambda \pmod 2$; as well as the quantity $S = s - \lambda(s-1) = \begin{cases} s, & \text{if } \lambda = 0 \\ 1, & \text{if } \lambda = 1 \end{cases}$.

Taking into account the Fermat's equation $\left(a_1^2\right)^2 + \left(2^{2s} b_1^2\right)^2 = \left(c_1^2\right)^2$ as a Pythagoras equation, and applying the Pythagorician divisors Theorem, cf. Theorem 2.1. & Lemma 2.1.:

It has been possible to find 2 proper divisors d_b and d_b' of $2^s b_1$, defined by:

$d_b = gcd\left(2^s b_1, c-a\right) = 2^S b_2$ and $d_b' = 2^{s-S} b_2' = \frac{2^s B_2}{d_b}$, where

$\left(2^s B_2\right)^2 = gcd\left(b^2, c^2 - a^2\right) \Rightarrow d_b d_b' = 2^s B_2$.

And finally to determine a third odd proper divisor b_2'' of $2^s b_1$ by the equality $2^s b_1 = d_b d_b' b_2'' = 2^s b_2' b_2 b_2''$.

In particular $b_1 = b_2' b_2 b_2''$, where b_2', b_2, b_2'' are odd and pairwise prime.

And in addition: $\begin{cases} \text{if } \lambda = 0 : b_2'' \neq 1; \text{ and } b_2' \neq 1 \Rightarrow b_2 \text{ is a proper divisor of } b_1; \\ \text{if } \lambda = 1 : b_2'' \neq 1; \text{ and } b_2 \neq 1 \Rightarrow b_2' \text{ is a proper divisor of } b_1. \end{cases}$

We call Fermat's absolute *b*-divisor, the following pairs of integers:

$$\left(d_b, d_b'\right) = \left(gcd\left(b, c-a\right), \frac{2^s B_2}{d_b}\right).$$

These very particular divisors of the even term $2^s b_1$, verify the following Pythagorean equation:

$$\begin{cases} \left(d_b'^4\right)^2 + \left(2^{-3} d_b^4\right)^2 = \left(b_2''^2\right)^2, & \text{if } \lambda = 0; \\ \left(\left(2^{-1} d_b\right)^4\right)^2 + \left(2 d_b'^4\right)^2 = \left(b_2''^2\right)^2, & \text{if } \lambda = 1. \end{cases}$$

The central element being even, the others odd.

From this last Pythagorean equation, it suffices to use a parameterization for Pythagorician triplet, of the type: $(u^2-v^2, 2uv, u^2+v^2)$, with $v \equiv 0 \pmod 2$, $u > v$, and $gcd(u,v)=1$, then we deduce the existence of integers $\alpha_2, \beta_2, \gamma_2 \in 2\mathbb{N}+1$, and coprime, such that $\alpha_2^4 + (2^{s-1}\beta_2)^4 = \gamma_2^4$, which contradicts the minimality of *s*.

2.2. Notations and Reminders for Pythagoras and Fermat Equations

Let's consider:

$$a^2 + b^2 = c^2, \text{ the Pythagoras equation.} \quad (2.2)$$

And:

$$a^4 + b^4 = c^4, \text{ the Fermat's equation.} \quad (2.3)$$

Remark 2.1 *For equation* (2.3) *we could suppose that positive trivial solution exist, for at the end, proove the impossibility, of such solution.*

Then cf. [8] and [9] we have:

Proposition 2.1 *The set of all the solutions of the Pythagoras equation* $a^2 + b^2 = c^2$ *(resp. and eventually these of the Fermat's equation:* $a^4 + b^4 = c^4$*), is formed from the solutions generated by all the positive primitive solutions, and their associates.*

Remark 2.2 1) *Let* $(a, b \neq 0, c) \in \mathbb{N}^{*3}$ *checking* (2.2), *then* $(a,b,c) \equiv (\pm 1, 0, 1) \pmod 4$.

2) Let $(a, b \neq 0, c) \in \mathbb{N}^{*3}$ checking (2.3), then $(a,b,c) \equiv (\pm 1, 0, 1) \pmod 4$, with $c \neq \pm a + 2$, $b \neq 2^s$, and $a \neq p^k$ where *p* is an odd prime and $k \neq 0$. In particular $b = 2^s b_1$ with $s \geq 2$ and necessarily $b_1 \geq 3$ odd. (see Proposition 2.3. for the proof).

This in no way restricts the expression of the generality of the solutions of said equations, because (b,a,c) is also a solution called "associated with (a,b,c)", such that $(b,a,c) \equiv (0, \pm 1, 1) \pmod 4$.

Definition 2.1 *We now define the following sets cf.* [8]*:*

1) $\overline{T^+}$: The set of non-trivial, primitive and positive Pythagorician solutions of the type $(a, b = 2^s b_1, c) \equiv (\pm 1, 0, 1) \pmod 4$. This set is exactly the set of non-trivial, primitive and positive Pythagorician solutions.

2) $\overline{T}$: The set of Fermat equation' solutions $(F): a^4 + b^4 = c^4$, which are non-trivial, primitive and positives of type $(a, b = 2^s b_1, c) \equiv (\pm 1, 0, 1) \pmod 4$, with: $c \neq \pm a + 2$, $b \neq 2^s$, and $a \neq p^k$ where *p* is any odd prime with $k \neq 0$. This set is exactely the set of non-trivial, primitive and positive solutions of Fermat equation' solutions. cf. Proposition 2.3.

Let us recall cf. [3] [10], that (see also [8] for another important parametrization):

Proposition 2.2

$$\overline{T^+} = \left\{ (u^2 - v^2, 2uv, u^2 + v^2); u, v \in \mathbb{N}^*, u > v, u + v \equiv 1 \pmod 2 \text{ and } gcd(u,v) = 1 \right\}.$$

2.3. Pythagorician Divisors Theorem Applied to Fermat's Equation $a^4 + b^4 = c^4$

2.3.1. Pythagorician Divisors

Let's remind the Pythagorician divisors Theorem cf. [8]:

Theorem 2.1 *Let* $(1,0,1) \neq (a, b = 2^s b_1, c) \equiv (\pm 1, 0, 1)(\bmod 4)$, $s \geq 2$, b_1 *odd*, a, b, c *pairwise prime, then there are equivalences between the following propositions:*

$$\text{(1)}\quad a^2 + b^2 = c^2;\ \text{(ii)}\ \begin{cases} c - b = d^2 \\ c + b = d''^2 \end{cases};\ \text{(iii)}\ \begin{cases} c - a = \dfrac{e^2}{2} = \dfrac{\left(2^{S_0}\overline{e}\right)^2}{2} \\ c + a = 2e''^2 = 2\left(2^{s-S_0}\dfrac{b_1}{\overline{e}}\right)^2 \end{cases}.$$

Where in this Theorem, the notations are:

Definition 2.2 1) $\lambda_0 \in \{0,1\}$ *is defined by:* $\dfrac{c-a}{2} \equiv \lambda_0 (\bmod 2)$.

2) $\mathbb{N}^* \ni S_0 = s - \lambda_0 (s-1) = \begin{cases} s \geq 2, & \text{if } \lambda_0 = 0 \\ 1, & \text{otherwise} \end{cases}$.

3) The pythagorician divisors (d, d'') (resp. (e, e'')) are defined by:

$$\begin{cases} (d, d'') = \left(gcd(a, c-b), \dfrac{a}{d}\right), \text{ note that } a = dd''; \\ (e, e'') = \left(gcd(b, c-a), \dfrac{b}{e}\right) = \left(2^{S_0}\overline{e}, 2^{s-S_0}\dfrac{b_1}{\overline{e}}\right), \overline{e} \text{ odd, } e \text{ even, note that } b = ee''. \end{cases}$$

Remark 2.3 *As results:* 1) $e = 2^{S_0}\overline{e}$, $e'' = 2^{s-S_0}\dfrac{b_1}{\overline{e}}$;

2) $gcd(e, e'') = 2^{\lambda_0}$; $gcd(d, d'') = 1$ and $gcd\left(\overline{e}, \dfrac{b_1}{\overline{e}}\right) = 1$.

Remark 2.4 *Cf.* [8], *we get the important relations:*

$$\begin{cases} d = -\dfrac{e}{2} + e'' = d'' - e = -d'' + 2e'' \\ d'' = \dfrac{e}{2} + e'' = d + e = -d + 2e'' \end{cases} \text{ and } \begin{cases} e = 2(d'' - e'') \\ e'' = \dfrac{e}{2} + d \end{cases} \tag{2.4}$$

2.3.2. Some Important Results on Fermat's Equation $a^4 + b^4 = c^4$

Let's now particularly consider the Fermat equation: $a^4 + b^4 = c^4$, $abc \neq 0$; with goal to conclude that $\overline{\mathcal{T}} = \varnothing$; with $b = 2^s b_1$, $b_1 \in 2\mathbb{N} + 1$, where $b_1 \geq 1$ odd, and $s = v_2(b) \geq 2$ (cf. Proposition 2.3.(i)) put for the 2-adic valuation of *b*, and choosed among all solutions of (2.3) **such that $s \geq 2$ have minimal value**; moreover *a*, b_1 and *c* are odd.

We have the following Proposition.

Proposition 2.3 *Let* $(a, b = 2^s b_1 \neq 0, c) \in \mathbb{N}^{*3}$, *relatively pairwise prime*/ $a^4 + (2^s b_1)^4 = c^4$, *then* $(a, b, c) \in \overline{\mathcal{T}}$ *cf. Definition* 2.1.(2), *that means that:*

i) $(a, b, c) \equiv (\pm 1, 0, 1)(\bmod 4)$, in particular $s \geq 2$.

ii) $b \neq 2^s$ (i.e. $b_1 \neq 1$), and $1 < a \neq p^k$ where *p* is an odd prime.

iii) $c \neq \pm a + 2$.

Proof 1 (i) *Let* $a^4 + b^4 = c^4$ *took as in Lemma* $\Rightarrow (a^2, b^2, c^2) \in \overline{T^+}$. *Let's put*

$d = gcd\left(a^2, c^2 - b^2\right)$, *then:* $c^2 - b^2 = d^2 \Rightarrow \left(d, b, c\right) \in \overline{T^+} \Rightarrow$ *(cf. Proposition* 2.2*)* $\exists \ u > v \in \mathbb{N}^*$, $u + v \equiv 1\left(\bmod 2\right)$ *such that* $b = 2uv \Rightarrow 4/b$ *and in addition* $c = u^2 + v^2 \equiv 1\left(\bmod 4\right)$, *while* $a = u^2 - v^2 \equiv \pm 1\left(\bmod 4\right)$ *because a is odd.*

In conclusion: $\left(a, b, c\right) \equiv \left(\pm 1, 0, 1\right)\left(\bmod 4\right)$.

(ii) • Suppose that $b = 2^s$ and $s \geq 2$ i.e. $b_1 = 1$.

Let $\left(e, e''\right) = \left(gcd\left(b^2, c^2 - a^2\right), \dfrac{b^2}{e}\right)$. In this case, considering the Pythagoras equation:

$\left(a^2\right)^2 + \left(b^2\right)^2 = \left(c^2\right)^2$, it comes: $\dfrac{c^2 - a^2}{2} \equiv 0\left(\bmod 2\right)$, i.e $\lambda_0 = 0$ cf. Definition 2.2. (1). So according to [8] or Remark 2.3.(2) gives: $pgcd\left(e, e''\right) = 2^{\lambda_0} = 1$, therefore, since $b^2 = ee''$ and that e is even, then e'' is odd; but then $b = 2^s \Rightarrow e'' = 1$.

But Theorem 2.1 gives: $c^2 + a^2 = 2e''^2 = 2 \Rightarrow c = a = 1$, which is absurd because $\left(a, b, c\right) \in \overline{\mathcal{T}}$. So as stated: $b \neq 2^s$, $\forall s \geq 2$.

- Similarly, for the other result: suppose that $a = p^k$, $k \geq 1$, that is to say a power of an odd natural prime number *p*.

Let us put: $\left(d, d''\right) = \left(gcd\left(a^2, c^2 - b^2\right), \dfrac{a^2}{d}\right)$ according to Definition 2.2., then we have:

$a^2 = p^{2n} = dd''$ and from Proposition 2.4. (3), saying $gcd\left(d, d''\right) = 1$.

We deduce that: either $\begin{cases} d = p^{2n} \\ d'' = 1 \end{cases}$; either $\begin{cases} d = 1 \\ d'' = p^{2n} \end{cases}$.

Now the Theorem 2.1. $\Rightarrow \begin{cases} c^2 - b^2 = d^2; \\ c^2 + b^2 = d''^2. \end{cases}$

- In the first case: $c^2 + b^2 = 1 \Rightarrow \left(a, b, c\right) = \left(1, 0, 1\right) \notin \overline{\mathcal{T}}$, which is absdurd.
- In the second case: $c^2 - b^2 = 1 \Rightarrow c = 1$ and $b = 0$; which is absurd.

Thus: $a \neq p^k$, $\forall k \in \mathbb{N}^*$ where *p* prime.

<u>Conclusion</u>: Neither *a* nor *b* can be a power of a prime natural number.

(iii) • Suppose that $c = a + 2$.

Then $\exists \ u > v \in \mathbb{N}^*$, $u + v \equiv 1\left(\bmod 2\right)$ such that $\left(a + 2\right)^2 = u^2 + v^2$ and $a^2 = u^2 - v^2 \Rightarrow 2\left(a + 1\right) = v^2$, but then: $u^2 = \left(a + 1\right)^2 + 1$, that is impossible, since the difference of two squares cannot be equal to 1.

- Suppose that $c = -a + 2$. If $a = 1 \Rightarrow c = 1$, that is absurd; If $a \geq 3 \Rightarrow c < 0$ that is absurd.

As a consequence as claimed $c \neq \pm a + 2$.

See also an altenative proof in [9] pages 59-60-61.

Remark 2.5 *Subsequently, as already said, when* $\left(a, b, c\right) \in \overline{\mathcal{T}}$*, i.e.* $a^4 + \left(2^s b_1\right)^4 = c^4$*, then we have:* $c \neq \pm a + 2$; $b_1 \neq 1$ *and* $a \neq p^k$, $\forall k \in \mathbb{N}^*$, $\forall p$, *an odd prime.*

2.3.3. Direct Application of the Pythagorician Divisors Theorem to Fermat's Equation: $a^4 + b^4 = c^4$

We now directly apply the results of the Pythagorician divisors Theorem to

Fermat's equation: $a^4+b^4=c^4$, because $\left(a^2,b^2,c^2\right)\in\overline{\mathcal{T}^+}$:

Proposition 2.4 *Consider Fermat's equation:* $a^4+b^4=c^4$, *and let* $\left(a,b=2^s b_1,c\right)\in\overline{\mathcal{T}}$, *a non-trivial hypothetical solution, of it.*

Then: $\exists\ A_2,a_2''\in 2\mathbb{N}+1/\,gcd\left(A_2,a_2''\right)=1$ and $B_2,b_2''\in 2\mathbb{N}+1/\,gcd\left(B_2,b_2''\right)=1$, such that:

(i) $a=A_2a_2''$ and $b=\left(2^s B_2\right)b_2''$.

$$\text{(ii)}\ \begin{cases}\left(d,d''\right)=\left(gcd\left(a^2,c^2-b^2\right),\dfrac{a^2}{d}\right)=\left(A_2^2,\left(\dfrac{a}{A_2}\right)^2=a_2''^2\right);\\ \left(e,e''\right)=\left(gcd\left(b^2,c^2-a^2\right),\dfrac{b^2}{e}\right)=\left(\left(2^s B_2\right)^2,\left(\dfrac{b_1}{B_2}\right)^2=b_2''^2\right).\end{cases}\tag{2.5}$$

Proof 2 *Indeed Pythagorician triplet:* $\left(a^2,b^2=\left(2^s b_1\right)^2,c^2\right)$ *is a non-trivial, primitive and positive one. Its Pythagorician divisors are in this case (since* $\dfrac{c^2-a^2}{2}\equiv 0\left(\mathrm{mod}\,2\right)$, *i.e* $\lambda_0=0$*):*

$$\begin{cases}\left(d,d''\right)=\left(gcd\left(a^2,c^2-b^2\right),\dfrac{a^2}{d}\right);\\ \left(e,e''\right)=\left(gcd\left(b^2,c^2-a^2\right),\dfrac{b^2}{e}\right).\end{cases}\Rightarrow\begin{cases}a^2=dd'',\ gcd\left(d,d''\right)=1;\\ b^2=ee'',\ gcd\left(e,e''\right)=1.\end{cases}\ \text{because}$$

remark 2.3 (2) $\Rightarrow$

$$\begin{cases}\exists A_2,a_2''\in 2\mathbb{N}+1,\text{ such that}: d=A_2^2\text{ and }d''=a_2''^2;\\ \exists B_2,b_2''\in 2\mathbb{N}+1,\text{ such that}: e=\left(2^s B_2\right)^2\text{ and }e''=b_2''^2.\end{cases}\Rightarrow\begin{cases}a=A_2a_2'';\\ b=2^s b_1=\left(2^s B_2\right)b_2''.\end{cases}\Rightarrow$$

$$\begin{cases}\left(d,d''\right)=\left(gcd\left(a^2,c^2-b^2\right),\dfrac{a^2}{d}\right)=\left(A_2^2,\left(\dfrac{a}{A_2}\right)^2=a_2''^2\right);\\ \left(e,e''\right)=\left(gcd\left(b^2,c^2-a^2\right),\dfrac{b^2}{e}\right)=\left(\left(2^s B_2\right)^2,\left(\dfrac{b_1}{B_2}\right)^2=b_2''^2\right).\end{cases}\tag{2.6}$$

The application of Theorem 2.1. where in addition one takes into account the results of Proposition 2.4, gives the following corollary (the notations remaining unchanged).

Lemma 2.1 *Consider* $\left(a,b=2^s b_1,c\right)$ *a positive triplet. There are equivalences between the following 3 propositions:*

(i) $\left(a,b=2^s b_1,c\right)\in\overline{\mathcal{T}}$.

(ii) $\begin{cases}c^2-b^2=d^2=A_2^4\\ c^2+b^2=d''^2=a_2''^4\end{cases}$.

(iii) $\begin{cases}c^2-a^2=\dfrac{e^2}{2}=\dfrac{\left(2^s B_2\right)^4}{2}\\ c^2+a^2=2e''^2=2b_2''^4\end{cases}$.

Remark 2.6 1) *Note that:* $a=A_2a_2''$, $b=2^s b_1=B_2b_2''$.

2) Note that necessarily: $a_2''\neq 1$ and $b_2''\neq 1$ (otherwise $\left(a,b,c\right)=\left(1,0,1\right)$

would be trivial, which would be contradictory).

Proof 3 *It suffices to apply Theorem* 2.1 *to the Pythagorician triplet* $\left(a^2, b^2 = \left(2^s b_1\right)^2, c^2\right)$ *taking into account the formulas* (2.5).

- But, let us continue as we are looking through Theorem 2.2., for a new proof of our proof.

We give the following proposition, which we will need in the next paragraph:

Proposition 2.5 *The notations being the same as those of Lemma* 2.1.*, we have:* $1 \neq b_2'' \equiv 1 \pmod 4$.

Proof 4 *From Remark* 2.6.*:* $b_2'' \neq 1$.

For proving $b_2'' \equiv 1 \pmod 4$: Lemma 2.1. (ii) & (iii) implies that:

$$\begin{cases} c^2 = \dfrac{a_2''^4 + A_2^4}{2} = \left(\dfrac{a_2''^2 + A_2^2}{2}\right)^2 + \left(\dfrac{a_2''^2 - A_2^2}{2}\right)^2; \\ a^2 = \left(A_2 a_2''\right)^2 = \left(\dfrac{a_2''^2 + A_2^2}{2}\right)^2 - \left(\dfrac{a_2''^2 - A_2^2}{2}\right)^2. \end{cases} \quad \text{And:}$$

$$\begin{cases} c^2 = \left(b_2''^2\right)^2 + \left(\dfrac{\left(2^s B_2\right)^2}{2}\right)^2; \\ a^2 = \left(b_2''^2\right)^2 - \left(\dfrac{\left(2^s B_2\right)^2}{2}\right)^2. \end{cases}$$

But then : $c^2 + a^2 = 2\left(\dfrac{a_2''^2 + A_2^2}{2}\right)^2 = 2\left(b_2''^2\right)^2$; from where:

$$a_2''^2 + A_2^2 = 2b_2''^2 \Rightarrow \left(\frac{a_2'' + A_2}{2}\right)^2 + \left(\frac{a_2'' - A_2}{2}\right)^2 = b_2''^2 \Rightarrow b_2'' \equiv 1 \pmod 4.$$

2.3.4. Other Classical Demonstrations, Using Well-Known Diophantine Equations of Degree 4

Remark 2.7 *Taking account Remarks* 2.4. *&* 2.6.*, as well as Propositions* 2.3.. *&* 2.5*, it is easy to give all the list of all the famous well-known diophantine equations of degree four, connected with the impossible resolution of* $a^4 + b^4 = c^4$*, equation whose these coefficients would be solutions; however, all of them admit only trivial solutions (or other are impossible), and if solution are not trivials, this would imply that* $a_2'' = 1$ *or* $b_2'' = 1$*, which would be absurd.*

All this would allow us to conclude right now, that $\overline{\mathcal{T}} = \varnothing$.

Remark 2.8 *As an exemple, because it will be to long to expose all this equations, we give the following: Applying Remark* 2.4.(i) *to* $\left(a^2\right)^2 + \left(b^2\right)^2 = \left(c^2\right)^2$*, in relation to Proposition* 2.4. *above, it comes that Pythagorician triplet* $\left(u^2 - v^2, 2uv, u^2 + v^2\right) \in T^+$*, is such that (cf. Remark* 2.4.*):*

$u = e'' = b_2''^2$ *and* $v = \dfrac{e}{2} = \dfrac{\left(2^s B_2\right)^2}{2}$*, which implies that:*

$$u^2 + v^2 = c^2 = \left(b_2''^2\right)^2 + \left(\frac{\left(2^s B_2\right)^2}{2}\right)^2,$$

But this means that the area of this right-angled triangle is the square: $\left(2^{s-1}B_2b_2''\right)^2$;

Wich is absurd, because that is the famous impossible squaring of the right-angled triangle area problem or 20th Diophantus Problem, proved by Fermat's result (cf. also [11] for a new proof).

This equation which is also (because $s \geq 2$) of the type: $x^4 + 4y^4 = z^2$ (i.e. an Euler equation) and does not admit solutions in integers, such that all of which are different from zero (cf. [12] p. 70, Exercice 1 or else [13] p. 38).

2.4. Concept of Absolute Fermat Divisors

We are now going to specify what are the integers $\left(b_2, b_2', b_2''\right)$ and $\left(a_2, a_2', a_2''\right)$, which appeared naturally, during factorizations. For that, we will introduce the notion of: "*Absolute Fermat's divisors*".

Consider first the following lemma:

Lemma 2.2 *let* $\left(a, b = 2^s b_1, c\right) \in \mathcal{T}$, *a non-trivial hypothetical solution of Fermat's equation:*

$a^4 + b^4 = c^4$. Recall that $d = gcd\left(a^2, c^2 - b^2\right) = A_2^2$, and that $e = gcd\left(b^2, c^2 - a^2\right) = \left(2^s B_2\right)^2$.

Let's set $d_a = gcd\left(a, c - b\right)$ and $d_b = gcd\left(b, c - a\right)$, then:

d_a divides A_2 and d_b divides $2^s B_2$, and so we can set: $d_a' = \dfrac{A_2}{d_a} \in \mathbb{N}^*$ and $d_b' = \dfrac{2^s B_2}{d_b} \in \mathbb{N}^*$.

Proof 5 *We use parts of proof of Theorem* 2.2. *hereafter, and the fact that* $\mathbb{Q}$ *is algebraically closed. Indeed:*

- $d_a' = \dfrac{A_2}{d_a}$ is necessarily an integer because (cf. proof of Theorem 2.2.):

  $c + b = d_a'^4 \Rightarrow d_a' \in \mathbb{Q}$ and d_a' is a rational root of $X^4 - (c + b) \in \mathbb{Z}[X] \Rightarrow d_a' \in \mathbb{N}^*$.

- Similarly $d_b' = \dfrac{2^s B_2}{d_b}$ is an integer too (i.e. $d_b / 2^s B_2$) because:

  Let $\lambda \in \{0,1\}$ such that $\dfrac{c-a}{2} \equiv \lambda \pmod 2$, this implies that $\dfrac{c+a}{2} \equiv 1 - \lambda \pmod 2$, consequently $\dfrac{c+a}{2^{1+\lambda}} \in \mathbb{N}^*$, but (cf. proof of Theorem 2.2) hereafter, d_b' is a rational root of $X^4 - \dfrac{c-a}{2^{1+\lambda}} \in \mathbb{Z}[X] \Rightarrow d_b' \in \mathbb{N}^*$.

This allows us to define the absolute Fermat divisors.

2.4.1. Definition of Absolute Fermat Divisors

Definition 2.3 *Consider Fermat's equation:* $a^4 + b^4 = c^4$, *and let* $\left(a, b = 2^s b_1, c\right) \in \mathcal{T}$, *be a not trivial eventual solution of it. We call Fermat's absolute a-divisor (resp. Fermat's absolute b-divisor) the following pairs of* *integers:* $(d_a, d'_a) = \left(gcd(a, c-b), \frac{A_2}{d_a}\right)$ (*resp.* $(d_b, d'_b) = \left(gcd(b, c-a), \frac{2^s B_2}{d_b}\right)$.

d'_a is said to be the co-adjunct divisor of d_a and vice versa. We have the same notion for d_b with respect to d'_b.

Definition 2.4 *Let* $(a, b = 2^s b_1, c) \in \mathcal{T}$*, then we set* $\lambda \in \{0,1\}$ *such that* $\frac{c-a}{2} \equiv \lambda (\bmod 2)$, *and* $S = s - \lambda(s-1)$. *Note that* $s - S = \lambda(s-1)$.

Proposition 2.6 *Let* $(a, b = 2^s b_1, c) \in \mathcal{T}$. *Then:*

1) There exists $b'_2, b_2 \in 2\mathbb{N}+1$ with $gcd(b'_2, b_2) = 1$, such that: $d_b = 2^S b_2$ and $d'_b = 2^{s-S} b'_2$. Then: $d_b \equiv 0 (\bmod 2)$ and $gcd(d_b, d'_b) = 2^\lambda$ and $B_2 = b'_2 b_2$.

2) $\exists$ $b''_2 = \frac{b_1}{B_2} \in 2\mathbb{N}+1$ (cf. formula (2.5))/ $b_1 = b'_2 b_2 b''_2$.

In addition $b''_2 \neq 1$; and $b'_2 \neq 1$ if $\lambda = 0$, while $b''_2 \neq 1$; $b_2 \neq 1$ if $\lambda = 1$. So b_2 (resp. b'_2) is a proper divisor of b_1.

Proof 6 *See parts of proof of Theorem* 2.2., *in particular points around formula* (2.7), *and also* (2.5) *and*(2.6).

Remark 2.9 *Then we'll set* $(d_a, d'_a) = (a_2, a'_2)$ *and so* $a_2 a'_2 = A_2$ *cf. Definition* 2.3. *Then (see the proof of Theorem* 2.2.*):*

$gcd(d_a, d'_a) = gcd(a_2, a'_2) = 1$ *and* $a'_2 a_2 a''_2 = a$, *where* $a''_2 = \frac{a}{A_2}$ *cf. Remark* 2.6.(1).

2.4.2. Fermat's Absolute Divisors Theorem

The notations being unchanged, we get "Fermat's absolute divisors Theorem":

Theorem 2.2 ***((First form) of Fermat's absolute divisors).***

There is equivalence between the following propositions:

(i) $(a, b = 2^s b_1, c) \in \mathcal{T}$ i.e. $(F): a^4 + b^4 = c^4$ is realized.

(ii) $\exists$ $(a, b_1, c) \in (2\mathbb{N}+1)^3$, coprime in pairs, for wich: $\exists$ $(b'_2, b_2, b''_2) \in (2\mathbb{N}+1)^3$ coprime in pairs, and $2 \le s \in \mathbb{N}$, checking $b_1 = b'_2 b_2 b''_2$, and such that for notations: $S = s - \lambda(s-1)$, with $\lambda \in \{0,1\}$ defined by $\frac{c-a}{2} \equiv \lambda (\bmod 2)$, $d_b = gcd(2^s b_1, c-a) = 2^S b_2$ and $d'_b = 2^{s-S} b'_2 = \frac{2^s B_2}{d_b}$, where $(2^s B_2)^2 = gcd(b^2, c^2 - a^2)$, then the following system is checked:

$$\begin{cases} c - a = \frac{d_b^4}{2^{2+\lambda}} = 2^{2-\lambda}\left(2^{S-1} b_2\right)^4; \\ c + a = 2^{1+\lambda} d_b'^4 = 2^{1+\lambda}\left(2^{s-S} b'_2\right)^4; \\ c^2 + a^2 = 2b_2''^4. \end{cases}$$

Remark 2.10 *Note that equations* (i) *and* (ii) *are equivalent to* (iii).

(iii) $\exists$ $(a'_2 \neq 1, a_2, a''_2 \neq 1) \in (2\mathbb{N}+1)^3$ odds, coprime in pairs, such that: $a = a'_2 a_2 a''_2$ and:

$$\begin{cases} c-b=d_a^4=a_2^4 \\ c+b=d_a'^4=a_2'^4 \\ c^2+b^2=a_2''^4 \end{cases} \Leftrightarrow \begin{cases} b=\dfrac{a_2'^4-a_2^4}{2} \\ c=\dfrac{a_2'^4+a_2^4}{2} \\ a_2'^8+a_2^8=2a_2''^4 \end{cases}.$$

And that in this case: $a_2' \neq 1$ and $a_2'' \neq 1$ otherwise $c=1$ and $b=0$, which would be absurd.

As a result a_2, a_2', a_2'' are proper divisors of $a=a_2'a_2a_2''$.

Remark 2.11 *Note that cf. Remark* 2.6. *and Propositions* 2.3. *&* 2.5, *we get:* $(b_2' \neq 1, b_2, b_2'' \neq 1)$ *when* $\lambda=0$, *(resp.* $(b_2 \neq 1, b_2', b_2'' \neq 1)$ *when* $\lambda=1$*) and they are all odds, relatively pairwise prime, and are respectively proper divisors of* b_1, *since we have the factorizations:* $b_1=b_2'b_2b_2''$; *as* $b=2^s b_1$.

Reminders 2.1 $\forall a,c \in \mathbb{N}^*$, $\forall n \in \mathbb{N}$, $n \geq 2$, *consider* $c^n-a^n=(c-a)\times T_n(c,a)$, where $T_n(c,a)=\sum_{k=0}^{n-1} c^{n-1-k}a^k$. Then: $gcd(c-a, T_n(c,a))=gcd(n, c-a)$.

In particular $gcd(c-a, T_4(c,a)) \in \{1,2,4\}$.

Let's now show this theorem.

Proof 7

- *The equivalences between the systems, within points* ii) *and* iii) *are obvious.*
- Moreover, it is clear that (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i).
- It therefore remains to show that: (i) $\Rightarrow$ (ii) (and that (i) $\Rightarrow$ (iii) What we will not do here, for the sake of simplifying the results).

Note that from: $b^4=c^4-a^4=(c-a)T_4(c,a)$, we get:

$$\delta_b' = gcd(c-a, T_4(c,a)) = gcd(4, c-a) = \begin{cases} 4, \text{ when } \dfrac{c-a}{2} \equiv 0 \pmod 2, (\text{i.e. } \lambda=0) \\ 2, \text{ when } \dfrac{c-a}{2} \equiv 1 \pmod 2, (\text{i.e. } \lambda=1) \end{cases}$$

- <u>Let us show that i)⇒ii)</u>. We know that $\delta_b' \in \{2,4\}$.

► 1) Case 1: $\delta_b'=4$ i.e $\lambda=0$.

$$b^4=(2^s b_1)^4=16(2^{s-1}b_1)^4=c^4-a^4=16\left(\frac{c-a}{4}\right)\left(\frac{T_4(c,a)}{4}\right).$$

From where:

$$(2^{s-1}b_1)^4=\left(\frac{c-a}{4}\right)\left(\frac{T_4(c,a)}{4}\right)=\frac{c-a}{4}\times\frac{c+a}{2}\times\frac{c^2+a^2}{2} \tag{2.7}$$

But: $gcd\left(\dfrac{c-a}{4}, \dfrac{T_4(c,a)}{4}\right)=1 \Rightarrow \dfrac{c-a}{4}=\beta_2^4$ and $\dfrac{T_4(c,a)}{4}=\beta_2^{*4}$ and β_2 even, β_2^* odd because $T_4(c,a)=\dfrac{c+a}{2}\times\dfrac{c^2+a^2}{2} \equiv 1 \pmod 2$

$$\Rightarrow \frac{c-a}{4}=\beta_2^4 \text{ and } \frac{T_4(c,a)}{4}=\beta_2^{*4}.$$

But then: $2^{s-1}b_1=\beta_2 \times \beta_2^* \Rightarrow b=2^s b_1=2\beta_2 \times \beta_2^*$ with $gcd(\beta_2, \beta_2^*)=1$, but: $c-a=4\beta_2^4$, therefore:

$d_b = gcd\left(b\ ,c-a\right) = gcd\left(2\beta_2 \times \beta_2^*, 4\beta_2^4\right) = 2\beta_2$. We therefore have: $\beta_2 = \frac{d_b}{2}$.

Let's put $\frac{c-a}{4} \times \frac{c+a}{2} = \left(2^{s-1} B_2\right)^4 \Rightarrow$

$$\begin{cases} c-a = 4\beta_2^4 = 4\left(\frac{d_b}{2}\right)^4 \\ c+a = \frac{\left(2^s B_2\right)^4}{2\left(c-a\right)} = \frac{\left(2^s B_2\right)^4}{\frac{d_b^4}{2}} = 2\left(\frac{2^s B_2}{d_b}\right)^4 = 2d_b'^4 \\ c^2+a^2 = \frac{b^4}{c^2-a^2} = 2\left(\frac{b}{2^s B_2}\right)^4 = 2b_2''^4 \end{cases}.$$

Consequently $\exists$ $b_2 = \beta_2$ and $b_2' = \frac{B_2}{b_2}$ odds such that:

$$gcd\left(b_2, b_2'\right) = 1, d_b = 2^s b_2, d_b' = b_2' \text{ and } gcd\left(d_b, d_b'\right) = 1 \tag{2.8}$$

Checking:

$$\begin{cases} c-a = \frac{d_b^4}{4} = 4\left(2^{s-1} b_2\right)^4 \\ c+a = 2d_b'^4 = 2b_2'^4 \\ c^2+a^2 = 2b_2''^4 \end{cases} \Rightarrow \text{ Point (iii) of the theorem, when } \lambda = 0.$$

As a result in this case:

$$((2.7) \Rightarrow b = 2^s b_2' b_2 b_2'') \Rightarrow b_1 = b_2' b_2 b_2'' \tag{2.9}$$

► 2) Case 2: $\delta_b' = 2$ i.e $\lambda = 1$.

$$b^4 = \left(2^s b_1\right)^4 = 16\left(2^{s-1} b_1\right)^4 = c^4 - a^4 = 4\left(\frac{c-a}{2}\right)\left(\frac{T_4\left(c,a\right)}{2}\right).$$

From where:

$$\left(2^{s-1} b_1\right)^4 = \left(\frac{c-a}{2}\right)\left(\frac{T_4\left(c,a\right)}{8}\right) = \frac{c-a}{2} \times \frac{c+a}{4} \times \frac{c^2+a^2}{2} \tag{2.10}$$

But: $\frac{c-a}{2} \equiv 1\left(\bmod 2\right)$ and

$$gcd\left(\frac{c-a}{2}, \frac{T_4\left(c,a\right)}{2}\right) = 1 \Rightarrow gcd\left(\frac{c-a}{2}, \frac{T_4\left(c,a\right)}{8}\right) = 1.$$

Which implies: $\frac{c-a}{2} = \beta_2^4$ and $\frac{T_4\left(c,a\right)}{8} = \beta_2^{*4}$,

with β_2 odd and β_2^* even, because $2^{s-1} b_1$ is even.

Then: $2^{s-1} b_1 = \beta_2 \beta_2^* \Rightarrow b = 2^s b_1 = 2\beta_2 \beta_2^*$. And so:

$$d_b = gcd\left(b, c-a\right) = gcd\left(2\beta_2\beta_2^*, 2\beta_2^4\right) = 2\beta_2 \Rightarrow \beta_2 = \frac{d_b}{2}.$$

Consequently: $c-a = \frac{d_b^4}{8} = 2\left(\frac{d_b}{2}\right)^4 \Rightarrow \frac{c-a}{2} = \left(\frac{d_b}{2}\right)^4$.

From where: $\exists$ $b_2 \equiv 1\left(\bmod 2\right)$ such that $d_b = 2b_2$, (here $b_2 = \beta_2$) moreover, considering $\frac{c^2-a^2}{8}=\left(2^{s-1}B_2\right)^4$, it comes:

$$c+a=\frac{8\left(2^{s-1}B_2\right)^4}{c-a}=\frac{\left(2^sB_2\right)^4}{2(c-a)}=\frac{8}{d_b^4}\frac{\left(2^sB_2\right)^4}{2}\Rightarrow\frac{c+a}{4}=\left(\frac{2^sB_2}{d_b}\right)^4=d_b'^4.$$

We then set: $b_2'=\frac{B_2}{b_2}\equiv 1(\operatorname{mod}2)$ which is quite an integer (cf. Lemma 2.1.).

Then: $d_b'=\frac{2^sB_2}{d_b}=2^{s-1}\frac{B_2}{b_2}\in\mathbb{N}\Rightarrow d_b'=2^{s-1}b_2'$.

In the end, we have the formulas of the Theorem 2.2.(iii) when $\lambda=1$:

$$\begin{cases} c-a=\frac{d_b^4}{8}=2\left(b_2\right)^4; \\ c+a=4\left(\frac{2^sB_2}{d_b}\right)^4=4\left(d_b'\right)^4=4\left(2^{s-1}b_2'\right)^4; \\ c^2+a^2=2b_2''^4. \end{cases}$$

Summaring, and considering (2.10), the quantities $b_2,b_2',b_2''\in 2\mathbb{N}+1$ verify:

$$\left(gcd\left(b_2,b_2'\right)=1,d_b=2b_2,d_b'=2^{s-1}b_2',gcd\left(d_b,d_b'\right)=2\right)\Rightarrow b_1=b_2b_2'b_2''. \ \blacksquare \quad (11)$$

3. New Proof of Fermat's Theorem for *n* = 4

We get the following corollary:

Corollary 3.1 *Let* $\left(a,b=2^sb_1,c\right)\in\overline{\mathcal{T}}$ *, and* $\lambda\in\{0,1\}$ *defined by:*

$$\frac{c-a}{2}\equiv\lambda(\operatorname{mod}2).$$

Then we have the proposition:

$\exists!$ $\left(b_{1-\lambda,2}\neq 1,b_{\lambda,2},b_2''\neq 1\right)\in\left(2\mathbb{N}+1\right)^3$, formed by strictly odd divisors of b_1, checking $b_{1-\lambda,2}b_{\lambda,2}b_2''=b_1$, and such that the following equation of degree 8 holds:

$F_{\lambda,2}:\left(b_{1-\lambda,2}^4\right)^2+\left(2^{4s-3}b_{\lambda,2}^4\right)^2=\left(b_2''^2\right)^2$, where: $b_{1-\lambda,2}\neq 1$ and $1\neq b_2''\equiv 1(\operatorname{mod}4)$.

And: $\left(b_{1-\lambda,2},b_{\lambda,2},b_2''\right)=\begin{cases}\left(b_2',b_2,b_2''\right) & \text{if } \lambda=0; \\ \left(b_2,b_2',b_2''\right) & \text{if } \lambda=1.\end{cases}$

The proof is obvious, it suffice to calculate $c^2+a^2=2b_2''^4$, from the values c and a extrated from Theorem 2.2.

Remark 3.1 1) *The degree* 8 *equation:* $\left(b_{1-\lambda,2}^4\right)^2+\left(2^{4s-3}b_{\lambda,2}^4\right)^2=\left(b_2''^2\right)^2$, *is of the type of that of* [7], *but ours is more precise, both at the level of the exponent "s", which is the key to our proof, as well as the other quantities which are in fact proper divisors of the odd part of b, and whose product gives back this same odd part.*

2) At this stage, we could also conclude in a classic way, because we find a right triangle whose area is equal to the square $\left(2^{s-1}b_2b_2'\right)^4$ which is impossible.

From this we deduce the goal of our paper:

Theorem 3.1 *Fermat's equation:* $a^4+b^4=c^4$, *does not admit solutions in* $\overline{\mathcal{T}}$ *(i.e.* $\overline{\mathcal{T}}=\varnothing$ *). Consequently its only solutions are those generated by the trivial* *one* $(1,0,1)$ *and its associates.*

Remark 3.2 *From Theorem 2.2. and Corollary 3.1., other new and original proofs exist, but the following giving is the most natural, and uses the argument of the infinite descent or the minimality of a parameter, between 2 similar Fermat equations.*

Proof 8 *Consider* $(a_1, b = 2^s b_1, c_1) \in \overline{\mathcal{T}}$ *, with* $s \geq 2$ *and minimal in all solutions of* $\overline{\mathcal{T}}$ *.*

Clearly $\left(b_{1-\lambda,2}^4, 2^{4s-3} b_{\lambda,2}^4, b_2''^2\right) \in \overline{T^+} \Rightarrow \exists u, v \in \mathbb{N}^*$, $u \equiv 1 (\operatorname{mod} 2)$, $v \equiv 0 (\operatorname{mod} 2)$, $u > v$, and $gcd(u,v) = 1$, checking:

$$\begin{cases} b_{1-\lambda,2}^4 = u^2 - v^2 \\ 2^{4s-3} b_{\lambda,2}^4 = 2uv \Rightarrow \exists \alpha_2, \beta_2 \in 2\mathbb{N}+1 \text{, and coprime, such that:} \\ b_2''^2 = u^2 + v^2 \end{cases}$$

$$\begin{cases} b_{1-\lambda,2}^4 = (u+v)(u-v) = \left(\alpha_2^4 + \left(2^{s-1}\beta_2\right)^4\right)\left(\alpha_2^4 - \left(2^{s-1}\beta_2\right)^4\right) \\ u = \alpha_2^4 ; v = \left(2^{s-1}\beta_2\right)^4 \text{ and } \alpha_2\beta_2 = b_{\lambda,2} \end{cases} \Rightarrow$$

$\exists$ $\alpha_2', \gamma_2 \in 2\mathbb{N}+1$, and coprime such that $\begin{cases} \gamma_2^4 = \alpha_2^4 + \left(2^{s-1}\beta_2\right)^4 \\ \alpha_2^4 = \alpha_2'^4 + \left(2^{s-1}\beta_2\right)^4 \end{cases}$, and $\alpha_2'\gamma_2 = b_{1-\lambda,2}$.

But these last 2 Fermat's equations, contradict the minimality of *s*: that is absurd.

Remark 3.3 *About the proof of Theorem* 3.1.

An alternative proof, which is a very particular recurrence, is to consider for fixed *k*, $k \in \mathbb{N}$, the following property:

$(\mathcal{P}_k)$: "The equation $a^4 + \left(2^k b_1\right)^4 = c^4$, with a, b_1, c odds; is not solvable".

It is clear that $(\mathcal{P}_0)$ is true and that $(\mathcal{P}_1)$ is true (cf. Proposition 2.3.(i)).

Let's make the following recurrence hypothesis:

"$(\mathcal{P}_k)$ is true from rank 0, up to rank $s-1$".

Let us deduce that $(\mathcal{P}_s)$ is true for $s \geq 2$, which means that $(\mathcal{P}_s)$ would be true $\forall k \in \mathbb{N}$.

Indeed, suppose the converse when $s \geq 2$, i.e. $(\mathcal{P}_s)$ is false. Then:

$\exists$ $a_1, b_1, c_1 \in 2\mathbb{N}+1$ such that $a_1^4 + \left(2^s b_1\right)^4 = c_1^4$.

But then, the previous demonstration shows precisely the existence of $\alpha_2, \beta_2, \gamma_2 \in 2\mathbb{N}+1$ such that:

$\alpha_2^4 + \left(2^{s-1}\beta_2\right)^4 = \gamma_2^4$ which means that $(\mathcal{P}_{s-1})$ is false, thus contradicting the recurrence hypothesis, that is absurd.

Remark 3.4 1) *Always about proof of Theorem* 3.1.

An other alternative proof, is to consider a solution: $a^4 + \left(2^s b_1\right)^4 = c^4$, i.e. $(a, b = 2^s b_1, c) \in \overline{\mathcal{T}}$ with $s \geq 2$ arbitrary (i.e. s not necessarily minimal) then after $s-1$ iterations as was done once in proof 8, we will have an equation of type $\alpha^4 + (2\beta)^4 = \gamma^4$, in odd integers, that is not solvable cf. Proposition 2.3.(i).

2) And what about the remaining, 8 degree equation coming from quantity *a*:

$a_2'^8 + a_2^8 = 2a_2''^4$? that extracted from the system of Remark 2.10., which follows Theorem 2.2, and which can be reduced to a Pythagorean equation or to an equation of the type [14], see also [8]. We do believe that a priori, it would also be able to produce an original and simple solution.

4. Conclusions and Perspectives

The use of Pythagorician divisors and Fermat's absolute divisors, will have allowed us to establish a simple and new method based on particular divisors of *b*, here exposed, for demonstrate the Fermat's great Theorem: $a^n + b^n = c^n$ when $n \equiv 0 \pmod 4$.

We do believe that this method can be used to bring something new in the Diophantine proof of Fermat's Theorem, when $n \equiv 2 \pmod 4$, i.e. $n = 2p$, p prime, cf. [15], and finally to the general case of Fermat's Theorem $a^p + b^p = c^p$, cf. [16] [17] and that, whether p / abc or not.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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