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On the two-variable generalized Laguerre polynomials

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Abstract: In this paper, we introduce the two variable generalized Laguerre polynomials (2VGLP) ${}_GL_n^{(\alpha,\beta)}(x,y)$. Some properties of these polynomials such as generating functions, summation formulae and expansions are also discussed.

Keywords: Laguerre polynomials, Jacobi polynomials, generating functions, summation formulae, expansions.

MSC: 33C05, 33C10, 33C15, 33C45.

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1. Introduction

The two variable Laguerre polynomials (2VLP) $L_n(x,y)$ are defined by the series (see [1–3]) as follows:

$$L_n(x,y) = n! \sum_{r=0}^{n} \frac{(-1)^r y^{n-r} x^r}{(n-r)!(r!)^2} \tag{1}$$

and specified by the following generating functions:

$$\sum_{n=0}^{\infty} L_n(x,y) t^n = (1-yt)^{-1} \exp\left(\frac{-xt}{1-yt}\right), \tag{2}$$

$$\sum_{n=0}^{\infty} \frac{L_n(x,y) t^n}{n!} = \exp(yt) C_0(xt), \tag{3}$$

where $C_0(x)$ denotes the 0^{th} order Tricomi function. The n^{th} order Tricomi function $C_n(x)$ is defined as [4]:

$$C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\, x^r}{r!(n+r)!}. \tag{4}$$

Also, the (2VLP) $L_n(x,y)$ satisfy the following properties:

$$L_n(x,y) = y^n L_n(x/y), \quad L_n(x,1) = L_n(x), \tag{5}$$

where $L_n(x)$ are the ordinary Laguerre polynomials [5]

$$L_n(x) = n! \sum_{r=0}^{n} \frac{(-1)^r x^r}{(n-r)!(r!)^2}. \tag{6}$$

The two variable associated Laguerre polynomials (2VALP) $L_n^{(\alpha)}(x,y)$ are defined by the series (see [1,6]) as follows:

$$L_n^{(\alpha)}(x,y) = \sum_{r=0}^{n} \frac{(-1)^r (1+\alpha)_n\, y^{n-r} x^r}{(1+\alpha)_r (n-r)!\, r!} \tag{7}$$

and specified by the following generating functions:

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x,y) t^n = (1-yt)^{-1-\alpha} \exp\left(\frac{-xt}{1-yt}\right), \tag{8}$$

$$\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(x,y)t^n}{(\alpha+1)_n}=\Gamma(\alpha+1)\exp(yt)C_\alpha(xt). \tag{9}$$

For $\alpha = 0$, equations (7), (8) and (9) reduces respectively to equations (1), (2) and (3). Also, the (2VALP) $L_n^{(\alpha)}(x,y)$ satisfy the following properties:

$$L_n^{(\alpha)}(x,y)=y^nL_n^{(\alpha)}(x/y),\quad L_n^{(\alpha)}(x,1)=L_n^{(\alpha)}(x), \tag{10}$$

where $L_n^{(\alpha)}(x)$ are the generalized Laguerre polynomials of one variable [5].

$$L_n^{(\alpha)}(x)=\sum_{r=0}^{n}\frac{(-1)^r(1+\alpha)_n\,x^r}{(1+\alpha)_r(n-r)!\,r!}. \tag{11}$$

Further, the Laguerre polynomials $L_n^{(\alpha)}(x)$ satisfy the following generating function [5]:

$$\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(x)t^n}{(1+\alpha)_n}=\exp(t)\,{}_0F_1[-;1+\alpha;-xt]. \tag{12}$$

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ [5] are define as:

$$P_n^{(\alpha,\beta)}(x)=\frac{(1+\alpha)_n}{n!}\,{}_2F_1\left[\begin{array}{ccc}-n,1+\alpha+\beta+n & ; & \frac{1-x}{2}\\ 1+\alpha & ; & \end{array}\right] \tag{13}$$

and specified by the following generating function:

$$\sum_{n=0}^{\infty}\frac{P_n^{(\alpha,\beta)}(x)t^n}{(\alpha+1)_n(\beta+1)_n}={}_0F_1(-;1+\alpha;\frac{1}{2}(x-1)t)\,{}_0F_1(-;1+\beta;\frac{1}{2}(x+1)t). \tag{14}$$

When $\alpha=\beta=0$ the polynomials (13) become the Legendre polynomials $P_n(x)$ [5]

$$P_n(x)={}_2F_1\left[\begin{array}{ccc}-n,n+1 & ; & \frac{1-x}{2}\\ 1 & ; & \end{array}\right]. \tag{15}$$

Ragab [7] defined the Laguerre polynomials of two variables $L^{(\alpha,\beta)}(x,y)$ as follows:

$$L_n^{(\alpha,\beta)}(x,y)=\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!}\sum_{r=0}^{n}\frac{(-y)^r\,L_{n-r}^{(\alpha)}(x)}{r!\,\Gamma(\alpha+n-r+1)\Gamma(\beta+r+1)}. \tag{16}$$

Chatterjea [8] obtained the following generating function for Ragab polynomials $L_n^{(\alpha,\beta)}(x,y)$:

$$\sum_{n=0}^{\infty}\frac{n!L_n^{(\alpha,\beta)}(x,y)t^n}{(\alpha+1)_n(\beta+1)_n}=e^t\,{}_0F_1(-;1+\alpha;-xt)\,{}_0F_1(-;1+\beta;-yt). \tag{17}$$

The aim of the present paper is to introduce the two variable generalized Laguerre polynomials (2VGLP) ${}_GL_n^{(\alpha,\beta)}(x,y)$ as a generalization of the above mentioned Laguerre polynomials $L_n(x,y)$ and $L_n^{(\alpha)}(x,y)$. Further, we discuss several properties of these polynomials such as generating functions, relationships with other polynomials, summation formulae and expansions of polynomials.

2. The two variable generalized Laguerre polynomials (2VGLP) ${}_GL_n^{(\alpha,\beta)}(x,y)$

In this section, we first define the two variable generalized Laguerre polynomials (2VGLP) ${}_GL_n^{(\alpha,\beta)}(x,y)$. Next, we present some generating functions for these polynomials.

Definition 1. The two variables generalized Laguerre polynomials (2VGLP) ${}_GL_n^{(\alpha,\beta)}(x,y)$ are defined as:

$$ {}_GL_n^{(\alpha,\beta)}(x,y) = \frac{(1+\alpha)_n(1+\beta)_n}{n!} \sum_{r=0}^{n} \frac{(-1)^r\, y^{n-r}x^r}{(1+\alpha+\beta)_r(n-r)!\, r!}. \tag{18} $$

Remark 1. 1. For $\alpha=\beta=0$ the (2VGLP) ${}_GL_n^{(\alpha,\beta)}(x,y)$ in (18) reduces to the (2VLP) $L(x,y)$.
2. For $\alpha=0$ *or* $\beta=0$ the (2VGLP) ${}_GL_n^{(\alpha,\beta)}(x,y)$ in (18) reduces to the (2VALP) $L_n^{(\alpha)}(x,y)$.

Theorem 2. *For the two variables generalized Laguerre polynomials (2VGLP) ${}_GL_n^{(\alpha,\beta)}(x,y)$. The following generating functions holds true:*

$$ \sum_{n=0}^{\infty} {}_GL_n^{(\alpha,\beta)}(x,y)t^n = \sum_{r=0}^{\infty} \frac{(1+\alpha)_r(1+\beta)_r(-xt)^r}{(1+\alpha+\beta)_r(r!)^2} {}_2F_1[1+\alpha+r, 1+\beta+r; 1+r; yt], \tag{19} $$

$$ \sum_{n=0}^{\infty} \frac{(c)_n\, n!\,{}_GL_n^{(\alpha,\beta)}(x,y)t^n}{(1+\alpha)_n(1+\beta)_n} = (1-yt)^{-c}{}_1F_1\left[\begin{array}{ccc} c & ; & \dfrac{-xt}{1-yt} \\ 1+\alpha+\beta & ; & \end{array}\right], \tag{20} $$

$$ \sum_{n=0}^{\infty} \frac{n!\,{}_GL_n^{(\alpha,\beta)}(x,y)t^n}{(1+\alpha)_n(1+\beta)_n} = \exp(yt)\,{}_0F_1[-; 1+\alpha+\beta; -xt]. \tag{21} $$

Proof of (19). Denoting the left hand side of (19) by L and using the definition (18), we get

$$ \begin{aligned} L &= \sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{(1+\alpha)_n(1+\beta)_n(-1)^r\, y^{n-r}x^r t^n}{(1+\alpha+\beta)_r\, n!(n-r)!\, r!} \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(1+\alpha)_{n+r}(1+\beta)_{n+r}(-1)^r\, y^n x^r t^{n+r}}{(1+\alpha+\beta)_r\, (n+r)!n!\, r!} \\ &= \sum_{r=0}^{\infty} \frac{(1+\alpha)_r(1+\beta)_r(-xt)^r}{(1+\alpha+\beta)_r(r!)^2} {}_2F_1[1+\alpha+r, 1+\beta+r; 1+r; yt]. \end{aligned} $$

This completes the proof of (19). □

Proof of (20). Denoting the left hand side of (20) by L and using the definition (18), we get

$$ \begin{aligned} L &= \sum_{n=0}^{\infty}\sum_{r=0}^{n} \frac{(c)_n(-1)^r\, y^{n-r}x^r t^n}{(1+\alpha+\beta)_r\, (n-r)!\, r!} \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(c)_{n+r}(-1)^r\, y^n x^r t^{n+r}}{(1+\alpha+\beta)_r\, n!\, r!} \\ &= \sum_{r=0}^{\infty} \frac{(c)_r(-xt)^r}{(1+\alpha+\beta)_r\, r!} \sum_{n=0}^{\infty} \frac{(c+r)_n(yt)^n}{n!} \\ &= \sum_{r=0}^{\infty} \frac{(c)_r(-xt)^r}{(1+\alpha+\beta)_r\, r!}(1-yt)^{-c-r} \\ &= (1-yt)^{-c}{}_1F_1\left[\begin{array}{ccc} c & ; & \dfrac{-xt}{1-yt} \\ 1+\alpha+\beta & ; & \end{array}\right]. \end{aligned} $$

This completes the proof of (20). Similarly, (21) can be proved. □

Special cases of (19),(20) and (21)

1. For $\alpha=\beta=0$ and $\beta=0$, (19) reduces respectively to (2) and (8).
2. For $\alpha=\beta=0$ and $\beta=0$, (21) reduces respectively to (3) and (9).
3. For $\alpha=\beta=0$ and $\beta=0$, (20) reduces respectively to the following well-known generating functions:

$$ \sum_{n=0}^{\infty} \frac{(c)_n L_n(x,y)t^n}{n!} = (1-yt)^{-c}{}_1F_1\left[c; 1; \frac{-xt}{1-yt}\right] \tag{22} $$

and

$$\sum_{n=0}^{\infty}\frac{(c)_n L_n^{(\alpha)}(x,y)t^n}{(1+\alpha)_n}=(1-yt)^{-c}{}_1F_1\left[c;1+\alpha;\frac{-xt}{1-yt}\right]. \tag{23}$$

4. Taking $c=\beta+1$ and $c=\alpha+1$ in equation (20) and using Kummer's first theorem [4]

$${}_1F_1[a;c;z]=e^z{}_1F_1[c-a;c;-z], \tag{24}$$

we get respectively the following generating functions:

$$\sum_{n=0}^{\infty}\frac{n!{}_GL_n^{(\alpha,\beta)}(x,y)t^n}{(1+\alpha)_n}=(1-yt)^{-\beta-1}\exp\left(\frac{-xt}{1-yt}\right){}_1F_1\left[\begin{array}{ccc}\alpha & ; & \frac{xt}{1-yt}\\ 1+\alpha+\beta & ; & \end{array}\right] \tag{25}$$

and

$$\sum_{n=0}^{\infty}\frac{n!{}_GL_n^{(\alpha,\beta)}(x,y)t^n}{(1+\beta)_n}=(1-yt)^{-\alpha-1}\exp\left(\frac{-xt}{1-yt}\right){}_1F_1\left[\begin{array}{ccc}\beta & ; & \frac{xt}{1-yt}\\ 1+\alpha+\beta & ; & \end{array}\right]. \tag{26}$$

5. Taking $c=\alpha+\beta+1$ in equation (20) and using (8), we have

$${}_GL_n^{(\alpha,\beta)}(x,y)=\frac{(1+\alpha)_n(1+\beta)_n}{n!(1+\alpha+\beta)_n}L_n^{(\alpha+\beta)}(x,y). \tag{27}$$

6. Replacing x by xy in equation (21) and using (12), we have

$${}_GL_n^{(\alpha,\beta)}(xy,y)=y^n\frac{(1+\alpha)_n(1+\beta)_n}{n!(1+\alpha+\beta)_n}L_n^{(\alpha+\beta)}(x). \tag{28}$$

3. Summation formulae

Theorem 3. *The following summation formulae for the (2VGLP) ${}_GL_n^{(\alpha,\beta)}(x,y)$ holds true:*

$${}_GL_n^{(\alpha,\beta)}(x,y)=\sum_{r=0}^{n}\frac{(1+\alpha)_n(\alpha)_r r!}{(1+\alpha)_r(1+\beta)_r n!}L_{n-r}^{(\beta-\alpha)}(x,y){}_GL_r^{(\alpha,\beta)}(-x,y), \tag{29}$$

$${}_GL_n^{(\alpha,\beta)}(x,y)=\sum_{r=0}^{n}\frac{(1+\alpha)_n(\alpha)_r}{(1+\alpha+\beta)_r n!}L_{n-r}^{(\beta-\alpha)}(x,y)L_r^{(\alpha+\beta)}(-x,y), \tag{30}$$

$${}_GL_n^{(\alpha,\beta)}(x,y)=\sum_{r=0}^{n}\frac{(1+\beta)_n(\beta)_r r!}{(1+\alpha)_r(1+\beta)_r n!}L_{n-r}^{(\alpha-\beta)}(x,y){}_GL_r^{(\alpha,\beta)}(-x,y), \tag{31}$$

$${}_GL_n^{(\alpha,\beta)}(x,y)=\sum_{r=0}^{n}\frac{(1+\beta)_n(\beta)_r}{(1+\alpha+\beta)_r n!}L_{n-r}^{(\alpha-\beta)}(x,y)L_r^{(\alpha+\beta)}(-x,y). \tag{32}$$

Proof of (29). From (25), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{n!{}_GL_n^{(\alpha,\beta)}(x,y)t^n}{(1+\alpha)_n}&=\left((1-yt)^{-(\beta-\alpha)-1}\exp\left(\frac{-xt}{1-yt}\right)\right)\\ &\times\left((1-yt)^{-\alpha}{}_1F_1\left[\begin{array}{ccc}\alpha & ; & -\frac{-xt}{1-yt}\\ 1+\alpha+\beta & ; & \end{array}\right]\right).\end{aligned} \tag{33}$$

Now, using (8) and (20), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{n!{}_GL_n^{(\alpha,\beta)}(x,y)t^n}{(1+\alpha)_n}&=\left(\sum_{n=0}^{\infty}L_n^{(\beta-\alpha)}(x,y)t^n\right)\left(\sum_{r=0}^{\infty}\frac{r!(\alpha)_r{}_GL_r^{(\alpha,\beta)}(-x,y)t^r}{(1+\alpha)_r(1+\beta)_r}\right)\\ &=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{(\alpha)_r\, r!}{(1+\alpha)_r(1+\beta)_r}L_{n-r}^{(\beta-\alpha)}(x,y){}_GL_r^{(\alpha,\beta)}(-x,y)t^n.\end{aligned} \tag{34}$$

Equating the coefficient of t^n on both sides of (34), we get the desired result (29) .
Similarly, we can obtain (30) by using (8) and (23) in (33). □

Proof of (31). From (26), we have

$$\sum_{n=0}^{\infty}\frac{n!{}_GL_n^{(\alpha,\beta)}(x,y)t^n}{(1+\beta)_n}=\left((1-yt)^{-(\alpha-\beta)-1}\exp\left(\frac{-xt}{1-yt}\right)\right)\times\left((1-yt)^{-\beta}{}_1F_1\left[\begin{array}{ccc}\beta & ; & -\dfrac{-xt}{1-yt}\\ 1+\alpha+\beta & ; & \end{array}\right]\right). \tag{35}$$

Now, using (8) and (20), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{n!{}_GL_n^{(\alpha,\beta)}(x,y)t^n}{(1+\beta)_n}&=\left(\sum_{n=0}^{\infty}L_n^{(\alpha-\beta)}(x,y)t^n\right)\left(\sum_{r=0}^{\infty}\frac{r!(\beta)_r{}_GL_r^{(\alpha,\beta)}(-x,y)t^r}{(1+\alpha)_r(1+\beta)_r}\right)\\&=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{(\beta)_r\,r!}{(1+\alpha)_r(1+\beta)_r}L_{n-r}^{(\alpha-\beta)}(x,y){}_GL_r^{(\alpha,\beta)}(-x,y)t^n.\end{aligned} \tag{36}$$

Equating the coefficient of t^n on both sides of (36), we get the desired result (31).
Similarly, we can obtain (32) by using (8) and (23) in (35). □

Remark 2.
1. For $\alpha=0$ or $\beta=0$, the results (29), (30), (31) and (32) reduces to summation formulae for the (2VALP) $L_n^{(\alpha)}(x,y)$.
2. For $\alpha=\beta=0$, the results (29), (30), (31) and (32) reduces to summation formulae for the (2VLP) $L_n(x,y)$.

4. Expansions of polynomials

Theorem 4. *The following expansions of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ in terms of the (2VGLP) ${}_GL_n^{(\alpha,\beta)}(x,y)$ holds true:*

$$\begin{aligned}P_n^{(\alpha+\lambda,\beta+\mu)}(x)&=\sum_{r=0}^{n}\frac{(n-r)!r!(\alpha+\lambda+1)_n(\beta+\mu+1)_n}{(1+\alpha)_{n-r}(1+\lambda)_{n-r}(1+\beta)_r(1+\mu)_r}\\&\times{}_GL_{n-r}^{(\alpha,\lambda)}(\frac{1}{2}(1-x),-y){}_GL_r^{(\beta,\mu)}(-\frac{1}{2}(1+x),-y),\end{aligned} \tag{37}$$

$$\begin{aligned}P_n^{(\alpha+\lambda,\beta+\mu)}(x)&=\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(n-r-s)!r!(1+\alpha+\lambda)_n(1+\beta+\mu)_n(-1)^s(y+z)^s}{(1+\alpha)_{n-r-s}(1+\lambda)_{n-r-s}(1+\beta)_r(1+\mu)_r\,s!}\\&\times{}_GL_{n-r-s}^{(\alpha,\lambda)}(\frac{1}{2}(1-x),y){}_GL_r^{(\beta,\mu)}(-\frac{1}{2}(1+x),z),\end{aligned} \tag{38}$$

$$\begin{aligned}P_n^{(\alpha+\lambda,\beta+\mu)}(x)&=\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(n-r-s)!(-y)^r(\frac{1}{2}(1+x))^s}{r!\,s!}\\&\times\frac{(1+\alpha+\lambda)_n(1+\beta+\mu)_n}{(1+\alpha)_{n-r-s}(1+\lambda)_{n-r-s}(1+\beta+\mu)_r}{}_GL_{n-r-s}^{(\alpha,\lambda)}(\frac{1}{2}(1-x),y).\end{aligned} \tag{39}$$

Proof of (37). Taking the generating function (14), replacing α and β by $\alpha+\lambda$ and $\beta+\mu$ respectively and using (21), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{P_n^{(\alpha+\lambda,\beta+\mu)}(x)t^n}{(\alpha+\lambda+1)_n(\beta+\mu+1)_n}&=\left(\sum_{n=0}^{\infty}\frac{n!{}_GL_n^{(\alpha,\lambda)}(\frac{1}{2}(1-x),y)t^n}{(1+\alpha)_n(1+\lambda)_n}\right)\left(\sum_{r=0}^{\infty}\frac{r!{}_GL_r^{(\beta,\mu)}(-\frac{1}{2}(1+x),-y)t^r}{(1+\beta)_r(1+\mu)_r}\right)\\&=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{(n-r)!r!{}_GL_{n-r}^{(\alpha,\lambda)}(\frac{1}{2}(1-x),y){}_GL_r^{(\beta,\mu)}(-\frac{1}{2}(1+x),-y)t^n}{(1+\alpha)_{n-r}(1+\lambda)_{n-r}(1+\beta)_r(1+\mu)_r}.\end{aligned}$$

Now, equating the coefficient of t^n from both sides, we get the desired result (37).
Similarly, we can prove (38) and (39). □

Remark 3. For $\lambda = \mu = 0$, the results (37), (38) and (39) reduces to the expansions of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ in terms of the (2VALP) $L_n^{(\alpha)}(x,y)$.

Remark 4. For $\alpha = \beta = \lambda = \mu = 0$, the results (37), (38) and (39) reduces to the following summation formulae for the classical Legendre polynomials $P_n(x)$ in terms of the (2VLP) $L_n(x,y)$:

$$P_n(x) = n!\sum_{r=0}^{n}\begin{pmatrix} n \\ r \end{pmatrix} L_{n-r}(\frac{1}{2}(1-x),y)L_r(-\frac{1}{2}(1+x),-y), \tag{40}$$

$$P_n(x) = n!\sum_{r=0}^{n}\sum_{s=0}^{n-r}\begin{pmatrix} n \\ r \end{pmatrix}\begin{pmatrix} n-r \\ s \end{pmatrix}(-1)^s(y+z)^s L_{n-r-s}(\frac{1}{2}(1-x),y)L_r(-\frac{1}{2}(1+x),z), \tag{41}$$

$$P_n(x) = n!\sum_{r=0}^{n}\sum_{s=0}^{n-r}\begin{pmatrix} n \\ r \end{pmatrix}\begin{pmatrix} n-r \\ s \end{pmatrix}\frac{(-y)^r(\frac{1}{2}(1+x))^s}{s!}L_{n-r-s}(\frac{1}{2}(1-x),y). \tag{42}$$

Remark 5. The results (40) and (41) are known results of Khan and Al-Gonah [9].

Theorem 5. *The following expansions of Ragab polynomials $L_n^{(\alpha,\beta)}(x,y)$ in terms of the (2VGLP) ${}_GL_n^{(\alpha\beta)}(x,y)$ holds true:*

$$\begin{aligned} L_n^{(\alpha+\lambda,\beta+\mu)}(x,y) =& \frac{(\alpha+\lambda+1)_n(\beta+\mu+1)_n}{n!} \\ &\times\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(n-r-s)!r!}{(1+\alpha)_{n-r-s}(1+\lambda)_{n-r-s}(1+\beta)_r(1+\mu)_r s!}{}_GL_{n-r-s}^{(\alpha,\lambda)}(x,y){}_GL_r^{(\beta,\mu)}(y,-y), \end{aligned} \tag{43}$$

$$\begin{aligned} L_n^{(\alpha+\lambda,\beta+\mu)}(x,y) =& \frac{(\alpha+\lambda+1)_n(\beta+\mu+1)_n}{n!} \\ &\times\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(n-r-s)!r!(-1)^s(x+y-1)^s}{(1+\alpha)_{n-r-s}(1+\lambda)_{n-r-s}(1+\beta)_r(1+\mu)_r s!}{}_GL_{n-r-s}^{(\alpha,\lambda)}(x,y){}_GL_r^{(\beta,\mu)}(y,x), \end{aligned} \tag{44}$$

$$\begin{aligned} L_n^{(\alpha+\lambda,\beta)}(x,y) =& \frac{(\alpha+\lambda+1)_n(\beta+1)_n}{n!} \\ &\times\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(n-r-s)!(-y)^s}{(1+\alpha)_{n-r-s}(1+\lambda)_{n-r-s}(1+\beta)_r s!}{}_GL_{n-r-s}^{(\alpha,\lambda)}(x,y)L_r^{(\beta)}(y), \end{aligned} \tag{45}$$

$$\begin{aligned} L_n^{(\alpha,\beta+\mu)}(x,y) =& \frac{(\alpha+1)_n(\beta+\mu+1)_n}{n!} \\ &\times\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(n-r-s)!(-x)^s}{(1+\beta)_{n-r-s}(1+\mu)_{n-r-s}(1+\alpha)_r s!}{}_GL_{n-r-s}^{(\beta,\mu)}(y,x)L_r^{(\alpha)}(x). \end{aligned} \tag{46}$$

Proof of (43). Taking the generating function (17), replacing α and β by $\alpha+\lambda$ and $\beta+\mu$ respectively and using (21), we have

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{n!L_n^{(\alpha+\lambda,\beta+\mu)}(x,y)t^n}{(\alpha+\lambda+1)_n(\beta+\mu+1)_n} &= \left(\sum_{n=0}^{\infty}\frac{n!{}_GL_n^{(\alpha,\lambda)}(x,y)t^n}{(1+\alpha)_n(1+\lambda)_n}\right)\left(\sum_{r=0}^{\infty}\frac{r!{}_GL_r^{(\beta,\mu)}(y,-y)t^r}{(1+\beta)_r(1+\mu)_r}\right)\left(\sum_{s=0}^{\infty}\frac{t^s}{s!}\right) \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(n-r-s)!r!{}_GL_{n-r-s}^{(\alpha,\lambda)}(x,y){}_GL_r^{(\beta,\mu)}(y,-y)t^n}{(1+\alpha)_{n-r-s}(1+\lambda)_{n-r-s}(1+\beta)_r(1+\mu)_r s!} \end{aligned}$$

Now, equating the coefficient of t^n from both sides, we get the desired result (43).
Similarly, we can prove (44), (45) and (46). □

Remark 6. For $\lambda = \mu = 0$, the results (43), (44), (45) and (46) reduces to the expansions of Ragab polynomials $L_n^{(\alpha,\beta)}(x,y)$ in terms of the (2VALP) $L_n^{(\alpha)}(x,y)$.

5. Conclusion

In this paper, the two variable generalized Laguerre polynomials (2VGLP) ${}_GL_n^{(\alpha,\beta)}(x,y)$ are introduced and certain properties of these polynomials are deduced. The results of this paper are important tool for discuss certain properties of other polynomials.

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