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Bounds for the Convex Combination of the First Seiffert and Logarithmic Means in Terms of Generalized Heronian Mean

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Abstract

In this paper, the bounds for convex combination of the first Seiffert and Logarithmic means by general Heronian mean

$$H_{p(\alpha)}(a,b) < \alpha P(a,b) + (1-\alpha)L(a,b) < H_{q(\alpha)}(a,b)$$

are proved, where $a, b > 0,\ a \neq b,\ \alpha \in [0,1]$. The left bound is the best possible.

Keywords: Seiffert mean; Heronian mean; Logarithmic mean; convex combination; bounds
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1 Introduction

In the paper [(15)], Gao, Guo and Li proved the following optimal inequalities:
Let $a, b > 0,\ a \neq b$ then

$$H_\delta(a,b) < P(a,b) \quad \text{for} \quad \delta \geq \pi - 2 \quad \text{and} \quad P(a,b) < H_\beta(a,b) \quad \text{for} \quad \beta \leq 1 \tag{1.1}$$

$$\text{and} \quad \delta = \pi - 2,\ \beta = 1 \quad \text{are the best constants,}$$

$$H_\gamma(a,b) < L(a,b) \quad \text{for} \quad \gamma = +\infty \quad \text{and} \quad L(a,b) < H_\tau(a,b) \quad \text{for} \quad \tau \leq 4 \tag{1.2}$$

$$\text{and} \quad \gamma = +\infty,\ \tau = 4 \quad \text{are the best constants.}$$

$P(a,b)$ is the first Seiffert mean introduced by Seiffert [(22)]

$$P(a,b) = \frac{a-b}{4\arctan\left(\sqrt{\frac{a}{b}}\right) - \pi} = \frac{a-b}{2\arcsin\left(\frac{a-b}{a+b}\right)} \quad \text{for} \quad a, b > 0,\ a \neq b. \tag{1.3}$$

$L(a,b)$ is the Logarithmic mean

$$L(a,b) = \frac{a-b}{\log a - \log b} \quad \text{for} \quad a, b > 0,\ a \neq b. \tag{1.4}$$

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$H_\omega(a,b)$ is the Heronian mean introduced by Janous [(18)]

$$H_\omega(a,b) = \frac{a+\omega\sqrt{ab}+b}{\omega+2} \quad \text{for} \quad 0 \le \omega < +\infty, \qquad (1.5)$$
$$= \sqrt{ab} \quad \text{for} \quad \omega = +\infty.$$

Recently, the Logarithmic, Seiffert and Heronian means have been the subject of intensive research. In particular, many remarkable inequalities for the means can be found in the literature [(1)-(28)].

In [(19)], Liu, Meng proved interesting bounds. They found the greatest values $\alpha_1,\ \alpha_2$ and the least values $\beta_1,\ \beta_2$ such that the double inequalities

$$\alpha_1 C(a,b) + (1-\alpha_1)G(a,b) < P(a,b) < \beta_1 C(a,b) + (1-\beta_1)G(a,b)$$

and

$$\alpha_2 C(a,b) + (1-\alpha_2)G(a,b) < P(a,b) < \beta_2 C(a,b) + (1-\beta_2)G(a,b)$$

hold for all $a,b>0$ with $a \ne b$, where $C(a,b)$, $G(a,b)$, $H(a,b)$ and $P(a,b)$ denote the Contraharmonic, Geometric, Harmonic, and Seifferts means of two positive numbers a and b, respectively.

Similarly, in [(28)], the following double inequality for $\alpha \in (0,1)$, was shown:

$$M_{\log 2/(\log 2-\log\alpha)}(a,b) \le \alpha A(a,b) + (1-\alpha)L(a,b) \le M_{(1+2\alpha)/3}(a,b)$$

holds for all $a,b>0$, each inequality becomes an equality if and only if $a=b$, and the given parameters $\log 2/(\log 2-\log\alpha)$ and $(1+2\alpha)/3$ in each inequality are best possible. Here $M_p(a,b)$, $L(a,b)$, $A(a,b)$ are Power, Logarithmic, Arithmetic means, respectively.

In [(21)], the following inequalities were proved. Let $\alpha \in (0,1/2)\cup(1/2,1),\ a \ne b,\ a,b>0$. Let $p(\alpha)$ be a solution of

$$\frac{1}{p}\log(1+p) + \log\alpha/2 = 0 \quad \text{in} \quad (-1,1).$$

Then

$$\text{if} \quad \alpha \in (0,1/2), \quad \text{then} \quad \alpha A(a,b) + (1-\alpha)G(a,b) < L_p(a,b) \quad \text{for} \quad p \ge p(\alpha)$$

and $p(\alpha)$ is the best constant,

$$\text{if} \quad \alpha \in (1/2,1), \quad \text{then} \quad \alpha A(a,b) + (1-\alpha)G(a,b) > L_p(a,b) \quad \text{for} \quad p \le p(\alpha)$$

and $p(\alpha)$ is the best constant.
Here $L_p(a,b), A(a,b), G(a,b)$ are Generalized logarithmic, Arithmetic, Geometric means, respectively.

It might be surprising that the means have applications in physics, economics, and even in meteorology. Logarithmic mean, which can be expressed in terms of Gausss's hypergeometric function ${}_2F_1$, has many applications. For example, a variant of Jensen's functional equation involving the Logarithmic mean, appears in heat conduction problem. Heronian and Seiffert means have applications in geometry, topology, fuzzy sets, ordinary differential equations and so on. For example, Runge-Kutta methods are based on the Heronian mean. Similarly, Seiffert mean is used for a characterization of Stolarsky means, which can been implicated in finding relative metrics.
It is well known, that $H_\omega(a,b)$ is a decreasing continuous function of the argument ω. From this and from results of Gao, Guo and Li [(15)] it follows that there exist optimal functions $p(\alpha), q(\alpha), 0 \le \alpha \le 1$ such that

$$H_{p(\alpha)}(a,b) < \alpha P(a,b) + (1-\alpha)L(a,b) < H_{q(\alpha)}(a,b).$$

Therefore, it is natural to ask what are the optimal functions. The purpose of this paper is to find or establish suitable bounds for the optimal functions. The inequalities we obtained are new and improve the existing corresponding results. The left inequality is optimal.

2 Main Results

In this section we prove the following theorem.

Theorem 2.1. *Let* $a, b > 0,\ a \neq b$. *Let* $P(a, b),\ L(a, b),\ H_\omega(a, b)$, *be the first Seiffert mean, the Logarithmic mean and the Heronian mean. Let* $\alpha \in [0, 1]$. *Then*

$$H_p(a, b) < \alpha P(a, b) + (1 - \alpha)L(a, b) \tag{2.1}$$

if and only if $p < p(\alpha) = \frac{\pi}{\alpha} - 2$.

If

$$\alpha P(a, b) + (1 - \alpha)L(a, b) < H_q(a, b) \tag{2.2}$$

then $q < 4 - 3\alpha$. *Moreover, if* $q \leq \frac{4}{1+3\alpha}$ *then (2.2) holds.*

Prof. (2.1), (2.2) are equivalent to

$$\frac{1}{b}H_p(a, b) < \frac{\alpha}{b}P(a, b) + \frac{(1 - \alpha)}{b}L(a, b) < \frac{1}{b}H_q(a, b). \tag{2.3}$$

Without loss of generality we can suppose that $0 < b < a$. Denote $t = \sqrt{b/a}$ and put

$$F(t, \alpha, \omega) = \alpha\frac{1 - t^2}{\pi - 4\arctan t} + (1 - \alpha)\frac{1 - t^2}{-2\ln t} - \frac{1}{\omega + 2}(t^2 + \omega t + 1). \tag{2.4}$$

Then $0 < t < 1$. First we prove the left inequality. This inequality is equivalent to $R(t, \alpha) = F(t, \alpha, \pi/\alpha - 2) > 0$ for $0 < t, \alpha < 1$. Some computation gives

$$R(t, \alpha) = \alpha\frac{1 - t^2}{\pi - 4\arctan t} + (1 - \alpha)\frac{1 - t^2}{-2\ln t} - \frac{\alpha t^2}{\pi} - \left(1 - \frac{2\alpha}{\pi}\right)t - \frac{\alpha}{\pi}. \tag{2.5}$$

Because of $R(t, \alpha)$ is a linear function of argument α, it suffices to show that $R(t, 0) > 0$ and $R(t, 1) > 0$ for $t \in (0, 1)$. Denote $F_1(t) = R(t, 0)$ and $F_2(t) = R(t, 1)$. Then

$$F_1(t) = \frac{s(t)}{2\ln t} = \frac{t^2 - 1 - 2t\ln t}{2\ln t}. \tag{2.6}$$

From $s''(t) = \frac{2t-2}{t} < 0$, $s'(1) = 0$, where $s'(t) = 2t - 2\ln t - 2$ we have $s'(t) > 0$ and from $s(1) = 0$ we obtain $s(t) < 0$ and so $R(t, 0) > 0$. $F_2(t) = R(t, 1) > 0$ is equivalent to

$$h(t) = \pi - 4\arctan t - \pi\frac{1 - t^2}{t^2 + (\pi - 2)t + 1} < 0. \tag{2.7}$$

Because of $h(0) = 0$, $h(1) = 0$ and $h'(0) = \pi^2 - 2\pi - 4 < 0$, where

$$h'(t) = -\frac{4}{1 + t^2} - \pi\left(\frac{-2t(t^2 + (\pi - 2)t + 1) - (1 - t^2)(2t + \pi - 2)}{(t^2 + (\pi - 2)t + 1)^2}\right) \tag{2.8}$$

it suffices to show that $h'(t)$ has only one root in $(0, 1)$. From $h'(t) = p(t)/((1+t^2)((t^2+(\pi-2)t+1)^2))$, where

$$p(t) = \left(\pi^2 - 2\pi - 4\right)t^4 + (16 - 4\pi)t^3 + \left(-2\pi^2 + 12\pi - 24\right)t^2 + (16 - 4\pi)t + \pi^2 - 2\pi - 4. \tag{2.9}$$

it suffices to show that $p(t)$ has only one root in $(0, 1)$.

Some computation gives that It is easy to show that

$$p(t) = (1 - t^2)q(t) = (1 - t^2)\left(\left(\pi^2 - 2\pi - 4\right)t^2 + \left(2\pi^2 - 8\pi + 8\right)t + \pi^2 - 2\pi - 4\right). \tag{2.10}$$

From $q(0) = \pi^2 - 2\pi - 4 < 0$ and $q(1) = 4\pi(\pi - 3) > 0$ we have $R(t, 1) > 0$.
From

$$\lim_{t \to 0^+} F(t, \alpha, \omega) = \frac{\alpha}{\pi} - \frac{1}{\omega + 2}$$

we obtain that $p(\alpha)$ is the best possible function.
Now we prove the right bounds.
To show $q(\alpha) > 4/(1 + 3\alpha)$ we need to prove $F(t, \alpha, 4/(1 + 3\alpha)) < 0$ for all $t, \alpha \in (0, 1)$. This is equivalent to

$$G(t, \alpha) = \alpha \frac{1 - t^2}{\pi - 4 \arctan t} + (1 - \alpha) \frac{1 - t^2}{-2 \ln t} - \frac{1 + 3\alpha}{6 + 6\alpha}(t^2 + \frac{4}{1 + 3\alpha} t + 1) < 0. \quad (2.11)$$

Some computation leads to

$$G''_{\alpha\alpha}(t, \alpha) = 6 \frac{1 - t^2}{(\pi - 4 \arctan t)(\ln t)} \left\{2 \ln t - 4 \arctan t + \pi\right\}. \quad (2.12)$$

Denote $u(t) = 2 \ln t - 4 \arctan t + \pi$. Because $u(1) = 0$ and

$$u'(t) = \frac{2t^2 - 4t + 2}{t(1 + t^2)} > 0$$

we have $u(t) < 0$ and $G''_{\alpha\alpha}(t, \alpha) > 0$. It implies $G(t, \alpha)$ is a convex function in argument α for each $t \in (0, 1)$. If we show that $G(t, 0) < 0$ and $G(t, 1) < 0$ for all $t \in (0, 1)$ then $G(t, \alpha) < 0$. $G(t, 0) < 0$ is equivalent to $s(t) = (3t^2 - 3)/(t^2 + 4t + 1) - \ln t > 0$. Simple computation leads to

$$s'(t) = \frac{\xi(t)}{t(t^2 + 4t + 1)^2} = \frac{-t^4 + 4t^3 - 6t^2 + 4t - 1}{t(t^2 + 4t + 1)^2}.$$

From $\xi'(t) = -4t^3 + 12t^2 - 12t + 4$, and $\xi''(t) = -12t^2 + 24t - 12 < 0$, $\xi'(1) = 0$, $\xi(1) = 0$ we have $\xi'(t) > 0$ and $\xi(t) < 0$. From this and from $s(1) = 0$ we obtain $G(t, 0) < 0$.
$G(t, 1) < 0$ is equivalent to $v(t) = 3(t^2 - 1)/(t^2 + t + 1) - 4 \arctan t + \pi > 0$. From $v(1) = 0$ it suffices to show that $v'(t) < 0$. It follows from

$$v'(t) = \frac{\xi(t)}{(1 + t^2)(t^2 + t + 1)^2} = \frac{-t^4 + 4t^3 - 6t^2 + 4t - 1}{(1 + t^2)(t^2 + 4t + 1)^2}.$$

To show $q(\alpha) < 4 - 3\alpha$ we need to prove $F(t, \alpha, 4 - 3\alpha) > 0$ for each α and some $t_\alpha \in (0, 1)$. Put $t_\alpha = 1 - \alpha + \alpha^2$. We show that $F(1 - \alpha + \alpha^2, \alpha, 4 - 3\alpha) > 0$ for $\alpha \in (0, 1)$. We use the following inequality

$$\frac{1 - t^2}{-\ln t} > \frac{1 - t^2}{\left(1 - t + \frac{(1-t)^2}{2} + \frac{(1-t)^3}{3t}\right)}.$$

This inequality follows from the Taylor's series for $\ln t$, $t \in (0, 1)$

$$-\ln t < 1 - t + \frac{(1 - t)^2}{2} + \frac{(1 - t)^3}{3} + \frac{(1 - t)^4}{3} + ...$$

To prove $F(t, \alpha, 4 - 3\alpha) > 0$ for $t_\alpha = 1 - \alpha + \alpha^2$, $\alpha \in (0, 1)$ it suffices to show that

$$(6\alpha - 3\alpha^2) \frac{1 - t_\alpha^2}{\pi - 4 \arctan t_\alpha} + (3\alpha^2 - 9\alpha + 6) \frac{1 - t_\alpha^2}{2\left(1 - t_\alpha + \frac{(1-t_\alpha)^2}{2} + \frac{(1-t_\alpha)^3}{3ta}\right)} \quad (2.13)$$

$$-t_\alpha^2 - (4 - 3\alpha)t_\alpha - 1 > 0.$$

Some calculation gives (17) is equivalent to

$$(6\alpha-3\alpha^2)\frac{1-t_\alpha^2}{\pi-4\arctan t_\alpha} > \frac{\varphi(t_\alpha)}{\psi(t_\alpha)}, \tag{2.14}$$

where

$$\varphi(t_\alpha)=t_\alpha^4-(1+3\alpha)t_\alpha^3+(9\alpha^2-12\alpha-3)t_\alpha^2+(9\alpha^2-21\alpha+5)t_\alpha-2, \tag{2.15}$$

$$\psi(t_\alpha)=t_\alpha^2-5t_\alpha-2. \tag{2.16}$$

It is easy to show that

$$t_\alpha^2=1-2\alpha+3\alpha^2-2\alpha^3+\alpha^4,$$

$$t_\alpha^3=1-3\alpha+6\alpha^2-7\alpha^3+6\alpha^4-3\alpha^5+\alpha^6,$$

$$t_\alpha^4=1-4\alpha+10\alpha^2-16\alpha^3+19\alpha^4-16\alpha^5+10\alpha^6-4\alpha^7+\alpha^8$$

$$\varphi(t_\alpha)=\alpha\chi\alpha)=\alpha(\alpha^7-7\alpha^6+27\alpha^5-61\alpha^4+91\alpha^3-105\alpha^2+72\alpha-36). \tag{2.17}$$

It implies

$$\chi(\alpha)<-36(1-\alpha)^2-46\alpha^2(1-\alpha)^2+\alpha^2(-23-15\alpha^2+27\alpha^3-7\alpha^4+\alpha^5). \tag{2.18}$$

Denote $k(\alpha)=-23-15\alpha^2+27\alpha^3-7\alpha^4+\alpha^5$. We have $k(\alpha)<l(u)=-23+12u-6u^2$, where $u=\alpha^2$ From $l(1)=-17$ and $l'(u)=12(1-u)>0$ we have $l(u)<0$ so $\varphi(t_\alpha)<0$. Evidently $\psi(t_\alpha)<0$. Denote

$$H(\alpha)=4\arctan(t_\alpha)-\pi+(6\alpha-3\alpha^2)(1-t_\alpha^2)\frac{\psi(t_\alpha)}{\varphi(t_\alpha)}. \tag{2.19}$$

We need to show $H(\alpha)>0$. From $H(1)=H(0)=0$ it suffices to show that $H'(\alpha)$ has only one root in $(0,1)$, and $H'(0)>0$. Some computation gives

$$H(\alpha)=4\arctan(1-\alpha+\alpha^2)-\pi- \tag{2.20}$$

$$\frac{(6-3\alpha)(12\alpha-24\alpha^2+25\alpha^3-14\alpha^4-\alpha^5+5\alpha^6-4\alpha^7+\alpha^8)}{-36+72\alpha-105\alpha^2+91\alpha^3-61\alpha^4+27\alpha^5-7\alpha^6+\alpha^7}.$$

This can be rewritten as

$$H(\alpha)=4\arctan(1-\alpha+\alpha^2)-\pi-\frac{a(\alpha)}{b(\alpha)}, \tag{2.21}$$

where

$$a(\alpha)=72\alpha-180\alpha^2+222\alpha^3-159\alpha^4+36\alpha^5+33\alpha^6-39\alpha^7+18\alpha^8-3\alpha^9, \tag{2.22}$$

$$b(\alpha)=-36+72\alpha-105\alpha^2+91\alpha^3-61\alpha^4+27\alpha^5-7\alpha^6+\alpha^7. \tag{2.23}$$

Some computation gives

$$H'(\alpha)=\frac{8\alpha-4}{2-2\alpha+3\alpha^2-2\alpha^3+\alpha^4}-\frac{a'(\alpha)b(\alpha)-a(\alpha)b'(\alpha)}{b^2(\alpha)}= \tag{2.24}$$

$$\frac{v(\alpha)}{(2-2\alpha+3\alpha^2-2\alpha^3+\alpha^4)b(\alpha)^2},$$

where

$$v(\alpha)=\alpha^3(\alpha-1)^3s(\alpha) \tag{2.25}$$

$$s(\alpha)=-2304+11232\alpha-26976\alpha^2+46372\alpha^3-59080\alpha^4+59711\alpha^5-48129\alpha^6+ \tag{2.26}$$

$$31498\alpha^7-16634\alpha^8+6980\alpha^9-2250\alpha^{10}+513\alpha^{11}-75\alpha^{12}+6\alpha^{13}.$$

We used

$$b^2(\alpha)=1296-5184\alpha+12744\alpha^2-21672\alpha^3+28521\alpha^4-29838\alpha^5+25483\alpha^6 \tag{2.27}$$

$$-17852\alpha^7+10249\alpha^8-4778\alpha^9+1765\alpha^{10}-500\alpha^{11}+103\alpha^{12}-14\alpha^{13}+\alpha^{14},$$

$$a'(\alpha)=72-360\alpha+666\alpha^2-636\alpha^3+180\alpha^4+198\alpha^5-273\alpha^6+144\alpha^7-27\alpha^8, \tag{2.28}$$

$$b'(\alpha)=72-210\alpha+273\alpha^2-244\alpha^3+135\alpha^4-42\alpha^5+7\alpha^6, \tag{2.29}$$

$$a'(\alpha)b(\alpha)=-2592+18144\alpha-57456\alpha^2+115200\alpha^3-159354\alpha^4+157122\alpha^5 \tag{2.30}$$

$$-103542\alpha^6+30120\alpha^7+24849\alpha^8-44007\alpha^9+36042\alpha^{10}-19818\alpha^{11}+7644\alpha^{12}$$

$$-2010\alpha^{13}+333\alpha^{14}-27\alpha^{15},$$

$$a(\alpha)b'(\alpha)=5184\alpha-28080\alpha^2+73440\alpha^3-124776\alpha^4+150228\alpha^5-130083\alpha^6+ \tag{2.31}$$

$$76920\alpha^7-22338\alpha^8-9603\alpha^9+16890\alpha^{10}-11610\alpha^{11}+5031\alpha^{12}-1434\alpha^{13}+252\alpha^{14}$$

$$-21\alpha^{15}.$$

Now, we show that $H'(\alpha)$ has only one root in $(0,1)$ which is equivalent to $s(\alpha)$ has only one root in $(0,1)$. Because $s(0)=-2304$ and $s(1)=864$ it suffices to show that $s'(\alpha)>0$

$$s'(\alpha)=11232-53952\alpha+139116\alpha^2-236320\alpha^3+298555\alpha^4-288774\alpha^5+ \tag{2.32}$$

$$220486\alpha^6-133072\alpha^7+62820\alpha^8-22500\alpha^9+5643\alpha^{10}-900\alpha^{11}+78\alpha^{12}.$$

Using $\alpha<1$ we get

$$s'(\alpha)>\xi(\alpha)=11232-53952\alpha+139116\alpha^2-236320\alpha^3+298555\alpha^4-288774\alpha^5+ \tag{2.33}$$

$$220486\alpha^6-133072\alpha^7+62820\alpha^8-22500\alpha^9+4743\alpha^{10}.$$

$$\frac{\xi(\alpha)}{10^5}>p(\alpha)=0.11-0.54\alpha+1.39\alpha^2-2.364\alpha^3+2.985\alpha^4-2.888\alpha^5+2.204\alpha^6 \tag{2.34}$$

$$-1.331\alpha^7+0.628\alpha^8-0.23\alpha^9+0.04\alpha^{10}.$$

Therefore, $s'(\alpha)>p(\alpha)$. Now we show $p(\alpha)>0$. We distinguish three cases.

Case 1. $\alpha\in(0,0.45]$.

Case 2. $\alpha\in[0.45,0.57]$.

Case 3. $\alpha\in[0.57,1)$.

Case 1. Denote

$$p_1(\alpha)=2.204\alpha^6-1.331\alpha^7+0.628\alpha^8-0.23\alpha^9+0.04\alpha^{10}. \tag{2.35}$$

$$p_2(\alpha)=0.11-0.54\alpha+1.39\alpha^2-2.364\alpha^3+2.985\alpha^4-2.888\alpha^5. \tag{2.36}$$

We have

$$p_1(\alpha)=\alpha^6q_1(\alpha)=\alpha^6(2.204-1.331\alpha+0.628\alpha^2-0.23\alpha^3+0.04\alpha^4). \tag{2.37}$$

Some computations give

$$q_1'(\alpha)=-1.331+1.256\alpha-0.69\alpha^2+0.16\alpha^3,\quad q_1''(\alpha)=1.256-1.38\alpha+0.48\alpha^2,$$

$$q_1'''(\alpha)=-1.38\alpha+0.96\alpha.$$

From $q_1'''(\alpha) < 0$ and $q_1''(1) = 0.356$ we have $q_1''(\alpha) > 0$. From this and from $q_1'(1) = -0.605$ we obtain $q_1'(\alpha) < 0$. Because $q_1(1) = 1.311$ we have $q_1(\alpha) > 0$ and so $p_1(\alpha) > 0$ for $\alpha \in (0, 1)$.

$$p_2'(\alpha) = -0.54 + 2.78\alpha - 7.092\alpha^2 + 11.94\alpha^3 - 14.44\alpha^4,$$

$$p_2''(\alpha) = 2.78 - 14.184\alpha + 35.82\alpha^2 - 57.76\alpha^3.$$

Using the Cardano's formula we obtain that $p_2''(\alpha) = 0$ only for one real $\alpha^* = 0.3218153$. From $p_2'(0) = -0.54$, $p_2'(\alpha^*) = -0.1367703$, $p_2'(0.45) = -0.2292278$ we get $p_2'(\alpha) < 0$ for $\alpha \in (0, 0.45]$. From $p_2(0.45) = 0.0021674$ we have $p(\alpha) = p_1(\alpha) + p_2(\alpha) > 0$ for $\alpha \in (0, 0.45]$.

Case 2. Put

$$q_1(\alpha) = p_1(\alpha) - 0.77\alpha^5, \quad q_2(\alpha) = p_2(\alpha) + 0.77\alpha^5 \tag{2.38}$$

We have

$$q_1(\alpha) = \alpha^5 r_1(\alpha) = \alpha^5 \left(-0.77 + 2.204\alpha - 1.331\alpha^2 + 0.628\alpha^3 - 0.23\alpha^4 + 0.04\alpha^5\right). \tag{2.39}$$

Some computations give

$$r_1'(\alpha) = 2.204 - 2.662\alpha + 1.884\alpha^2 - 0.92\alpha^3 + 0.2\alpha^4,$$

$$r_1''(\alpha) = -2.662 + 3.768\alpha - 2.76\alpha^2 + 0.8\alpha^3,$$

$$r_1'''(\alpha) = 3.768 - 5.52\alpha + 2.4\alpha^2, \quad r_1''''(\alpha) = -5.52 + 4.8\alpha < 0.$$

From $r_1''''(\alpha) < 0$ and $r_1'''(0.57) = 1.40136$ we have $r_1'''(\alpha) > 0$. From this and from $r_1''(0.57) = -1.2628096$ we obtain $r_1''(\alpha) < 0$. From $r_1'(0.57) = 1.149506$ we have $r_1'(\alpha) > 0$ for $\alpha \in (0.45, 0.57]$. From $r_1(0.45) = 0.0008057$ we have $q_1(\alpha) > 0$ for $\alpha \in (0.45, 0.57]$.

$$q_2'(\alpha) = -0.54 + 2.78\alpha - 7.092\alpha^2 + 11.94\alpha^3 - 10.59\alpha^4,$$

$$q_2''(\alpha) = 2.78 - 14.184\alpha + 35.82\alpha^2 - 42.36\alpha^3,$$

$$q_2'''(\alpha) = -14.184 + 71.64\alpha - 127.08\alpha^2 \quad q_2''''(\alpha) = 71.64 - 254.16\alpha < 0.$$

From $q_2'''(0.45) = -7.6797$ we have $q_2'''(\alpha) < 0$. From $q_2''(0.45) = -0.209305$ we have $q_2''(\alpha) < 0$. From $q_2'(0.45) = -0.0713537$ we get $q_2'(\alpha) < 0$. From $q_2(0.57) = 0.003673$ we have $p(\alpha) = q_1(\alpha) + q_2(\alpha) > 0$ for $\alpha \in [0.45, 0.57]$.

Case 3. Using elementary computations we obtain

$$p(1) = 0.004,\ p'(1) = -0.091,\ p''(1) = -0.918,\ p'''(1) = -6.966,$$

$$p^{(4)}(1) = -38.4,\ p^{(5)}(1) = -161.64,\ p^{(6)}(1) = -323.28,$$

$$p^{(7)}(1) = 1073.52,\ p^{(8)}(1) = 14434.56,\ p^{(9)}(1) = 61689.6,\ p^{(10)}(1) = 145152.$$

From this we obtain

$$p(\alpha) = \zeta(1-\alpha) = 0.004 + 0.091(1-\alpha) - 0.459(1-\alpha)^2 + 1.161(1-\alpha)^3 - 1.6(1-\alpha)^4 + \tag{2.40}$$

$$1.347(1-\alpha)^5 - 0.449(1-\alpha)^6 - 0.213(1-\alpha)^7 + 0.358(1-\alpha)^8 - 0.17(1-\alpha)^9 + 0.04(1-\alpha)^{10}.$$

Denote $s_\alpha = 1 - \alpha$. If we show $\zeta(s_\alpha) > 0$ for $0 < s_\alpha < 0.44$ then the proof will be completed. Denote

$$\zeta_1(s_\alpha) = 0.004 - 0.449s_\alpha^6 - 0.213s_\alpha^7 + 0.358s_\alpha^8 - 0.17s_\alpha^9 + 0.04s_\alpha^{10}. \tag{2.41}$$

$$\zeta_2(s_\alpha) = 0.091s_\alpha - 0.459s_\alpha^2 + 1.161s_\alpha^3 - 1.6s_\alpha^4 + 1.347s_\alpha^5. \tag{2.42}$$

We have

$$\zeta_1'(s_\alpha) = s_\alpha^5 \varrho_1(s_\alpha) = s_\alpha^5 \left(-2.694 - 1.491s_\alpha + 2.864s_\alpha^2 - 1.53s_\alpha^3 + 0.4s_\alpha^4\right). \tag{2.43}$$

Some computations give

$$\varrho_1'(s_\alpha) = -1.491 + 5.728s_\alpha - 4.59s_\alpha^2 + 1.6s_\alpha^3.$$

Using the Cardano's formula we obtain that $\varrho_1''(s_\alpha) = 0$ only for one real $s_\alpha^* = 0.3435535$. From $\varrho_1(0) = -2.694$, $\varrho_1(0.44) = -2.9109087$, $\varrho_1(s_\alpha^*) = -2.9246712$ we get $\zeta_1'(s_\alpha) < 0$. From $\zeta_1(0.44) = 0.0004706$ we have $\zeta_1(s_\alpha) > 0$ for $s_\alpha \in (0, 0.44]$. Next we have

$$\zeta_2(s_\alpha) = s_\alpha z(s_\alpha) = s_\alpha \left(0.091 - 0.459s_\alpha + 1.161s_\alpha^2 - 1.6s_\alpha^3 + 1.347s_\alpha^4\right),$$

$$z'(s_\alpha) = -0.459 + 2.322s_\alpha - 4.8s_\alpha^2 + 5.388s_\alpha^3.$$

Using the Cardano's formula we obtain that $z'(s_\alpha) = 0$ only for one real $s_\alpha^{**} = 0.3534804$. From $z(0) = 0.091$, $z(0.44) = 0.280021$, $z(s_\alpha^{**}) = 0.0241802$ we have $\zeta_2(s_\alpha) > 0$ for $s_\alpha \in (0, 0.44]$. The proof is complete.

Competing Interests

The author declares that he has no competing interests.

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