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Complete homogeneous symmetric functions of Gauss Fibonacci polynomials and bivariate Pell polynomials

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Abstract: In this paper, we introduce a symmetric function in order to derive a new generating functions of bivariate Pell Lucas polynomials. We define complete homogeneous symmetric functions and give generating functions for Gauss Fibonacci polynomials, Gauss Lucas polynomials, bivariate Fibonacci polynomials, bivariate Lucas polynomials, bivariate Jacobsthal polynomials and bivariate Jacobsthal Lucas polynomials.

Keywords: Symmetric functions, generating functions, Gauss Fibonacci polynomials, bivariate Pell polynomials, bivariate Jacobsthal polynomials.

MSC: 05E05, 11B39.

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1. Introduction

Özcan and Taştan defined the Gauss Fibonacci $GF_n(x)$ and Gauss Lucas $GL_n(x)$ polynomials and gave their Binet's formula [1]. The Gauss Fibonacci polynomials $\{GF_n(x)\}_{n\in\mathbb{N}}$ are defined by the following recurrence relation:

$$GF_{n+1}(x) = xGF_n(x) + GF_{n-1}(x), n \geq 2, \tag{1}$$

with initial conditions $GF_1(x) = 1$ and $GF_2(x) = x + i$.

The Gauss Lucas polynomials $\{GL_n(x)\}_{n\in\mathbb{N}}$ are defined by the following recurrence relation:

$$GL_{n+1}(x) = xGL_n(x) + GL_{n-1}(x), n \geq 2, \tag{2}$$

with initial conditions $GL_1(x) = x + 2i$ and $GL_2(x) = x^2 + 2 + ix$.

Now we can get the Binet's formula of Gauss Fibonacci and Gauss Lucas polynomials. Let $\alpha(x)$ and $\beta(x)$ be the solutions of the characteristic equation $t^2 - xt - 1 = 0$ of the recurrence relations (1) and (2). Then

$$\alpha(x) = \frac{x + \sqrt{x^2+4}}{2},\ \beta(x) = \frac{x - \sqrt{x^2+4}}{2}.$$

So we obtain

$$GF_n(x) = \frac{\alpha^{n-1}(x)\left(\alpha(x)+i\right) - \beta^{n-1}(x)\left(\beta(x)+i\right)}{\alpha(x) - \beta(x)},$$

and

$$GL_n(x) = \alpha^{n-1}(x)\left(\alpha(x)+i\right) + \beta^{n-1}(x)\left(\beta(x)+i\right).$$

The bivariate Pell $\{P_n(x,y)\}_{n\in\mathbb{N}}$ and bivariate Pell Lucas $\{Q_n(x,y)\}_{n\in\mathbb{N}}$ polynomials are defined by the following recurrence relations:

$$P_n(x,y) = 2xyP_{n-1}(x,y) + yP_{n-2}(x,y),\ n \geq 2,$$

with initial conditions $P_0(x,y) = 0$ and $P_1(x,y) = 1$.

$$Q_n(x,y) = 2xyQ_{n-1}(x,y) + yQ_{n-2}(x,y),\ n \geq 2,$$

with initial conditions $Q_0(x,y) = 2$ and $Q_1(x,y) = 2xy$.

Special cases of these bivariate polynomials are Pell polynomials $P_n(x,1)$, Pell Lucas polynomials $Q_n(x,1)$, Pell numbers $P_n(1,1)$ and Pell Lucas numbers $Q_n(1,1)$. The Binet's formula for bivariate Pell and bivariate Pell Lucas polynomials are given by

$$P_n(x,y) = \frac{\left(xy + \sqrt{x^2y^2+y}\right)^n - \left(xy - \sqrt{x^2y^2+y}\right)^n}{2\sqrt{x^2y^2+y}},$$

and

$$Q_n(x,y) = \left(xy + \sqrt{x^2y^2+y}\right)^n + \left(xy - \sqrt{x^2y^2+y}\right)^n,$$

respectively.

In 2018, Zorcelik and Uygun defined the bivariate Jacobsthal and bivariate Jacobsthal Lucas polynomials and gave Binet's formula of these polynomials [2].

Now, we define bivariate Jacobsthal polynomials, bivariate Jacobsthal Lucas polynomials, bivariate Fibonacci polynomials and bivariate Lucas polynomials.

Definition 1. For $n \in \mathbb{N}$, the bivariate Jacobsthal polynomials are defined by

$$J_n(x,y) = xyJ_{n-1}(x,y) + 2yJ_{n-2}(x,y) \text{ for } n \geq 2,$$

with initial conditions $J_0(x,y) = 0$ and $J_1(x,y) = 1$.

Definition 2. For $n \in \mathbb{N}$, the bivariate Jacobsthal Lucas polynomials, say $\{j_n(x,y)\}_{n\in\mathbb{N}}$ are defined by

$$j_n(x,y) = xyj_{n-1}(x,y) + 2yj_{n-2}(x,y) \text{ for } n \geq 2,$$

with initial conditions $j_0(x,y) = 2$ and $j_1(x,y) = xy$.

Definition 3. For $n \in \mathbb{N}$, the bivariate Fibonacci polynomials are defined by

$$F_n(x,y) = xF_{n-1}(x,y) + yF_{n-2}(x,y) \, for \, n \geq 2,$$

with initial conditions $F_0(x,y) = 0$ and $F_1(x,y) = 1$.

Definition 4. For $n \in \mathbb{N}$, the bivariate Lucas polynomials, say $\{L_n(x,y)\}_{n\in\mathbb{N}}$ are defined recurrently by

$$L_n(x,y) = xL_{n-1}(x,y) + yL_{n-2}(x,y) \text{ for } n \geq 2,$$

with initial conditions $L_0(x,y) = 2$ and $L_1(x,y) = x$.

In this contribution, we will define complete homogeneous symmetric function for which we can formulate, extend and prove results based on [3–5]. In order to determine generating functions of Gauss Fibonacci polynomials, Gauss Lucass polynomials, bivariate Pell polynomials, bivariate Pell Lucas polynomials, bivariate Fibonacci polynomials, bivariate Lucas polynomials, bivariate Jacobsthal polynomials and bivariate Jacobsthal Lucas polynomials, we use analytical means and series manipulation methods. In the sequel, we derive new symmetric functions and give some interesting properties. We also give some more useful definitions which are used in the subsequent sections. From these definitions, we prove our main results given in Section 3.

2. Preliminaries and definitions

In this section, we introduce symmetric function and give its properties [6–11].

Definition 5. Let A and B be any two alphabets, then we give $S_n(A-B)$ by the following form:

$$\frac{\prod_{b\in B}(1-zb)}{\prod_{a\in A}(1-za)} = \sum_{n=0}^{+\infty} S_n(A-B)z^n = \sigma_z(A-B). \tag{3}$$

with the condition $S_n(A-B)=0$ for $n<0$.

Remark 1. Taking $A=\{0\}$ in (3) gives

$$\Pi_{b\in B}(1-zb)=\sum_{n=0}^{+\infty}S_n(-B)z^n=\lambda_z(-B). \tag{4}$$

Further, in the case $A=\{0\}$ or $B=\{0\}$, we have

$$\sum_{n=0}^{+\infty}S_n(A-B)z^n=\sigma_z(A)\times\lambda_z(-B). \tag{5}$$

Thus,

$$S_n(A-B)=\sum_{k=0}^{n}S_{n-k}(A)S_k(-B).$$

Definition 6. Let k and n be two positive integers and $\{a_1,a_2,...,a_n\}$ are set of given variables. The k-th complete homogeneous symmetric function $h_k\left(a_1,a_2,...,a_n\right)$ is defined by

$$h_k\left(a_1,a_{2,}...,a_n\right)=\sum_{i_1+i_2+...+i_n=k}a_1^{i_1}a_2^{i_2}...a_n^{i_n},\ (0\leq k\leq n),$$

with $i_1,i_2,...,i_n\geq 0$.

Definition 7. [9] Let $A=\{a_1,a_2\}$ an alphabet. The complete homogeneous symmetric function $h_n\left(a_1,a_2\right)$ is defined by

$$h_n(a_1,a_2)=\frac{a_1^{n+1}-a_2^{n+1}}{a_1-a_2},\ n\in\mathbb{N}_0.$$

Definition 8. Let g be any function on $\mathbb{R}^n$, then we the divided difference operator is defined by

$$\partial_{x_ix_{i+1}}(g)=\frac{g(x_1,\cdots,x_i,x_{i+1},\cdots x_n)-g(x_1,\cdots x_{i-1},x_{i+1,}x_i,x_{i+2}\cdots x_n)}{x_i-x_{i+1}}.$$

3. Construction of generating functions of some polynomials

The following proposition is one of the key tools for proofing our main results. It has been proved in [9].

Proposition 1. *Consider an alphabet* $A=\{a_1,-a_2\}$ *, then*

$$\sum_{n=0}^{+\infty}h_n(a_1,[-a_2])z^n=\frac{1}{1-\left(a_1-a_2\right)z-a_1a_2z^2}. \tag{6}$$

Based on the relation (6) we have

$$\sum_{n=0}^{+\infty}h_{n-1}(a_1,[-a_2])z^n=\frac{z}{1-\left(a_1-a_2\right)z-a_1a_2z^2}. \tag{7}$$

Choosing a_1 and a_2 such that $\begin{cases}a_1-a_2=x\\ a_1a_2=1\end{cases}$ and substituting in (6) and (7), we obtain

$$\sum_{n=0}^{+\infty}h_n(a_1,[-a_2])z^n=\frac{1}{1-xz-z^2}, \tag{8}$$

$$\sum_{n=0}^{+\infty}h_{n-1}(a_1,[-a_2])z^n=\frac{z}{1-xz-z^2}, \tag{9}$$

respectively.

Multiplying Equation (9) by $(1+iz)$, we obtain

$$\sum_{n=0}^{+\infty}(1+iz)\,h_{n-1}(a_1,[-a_2])z^n=\frac{z+iz^2}{1-xz-z^2},$$

and we have the following theorem.

Theorem 1. *For $n\in\mathbb{N}$, the generating function of Gauss Fibonacci polynomials is given by*

$$\sum_{n=0}^{+\infty}GF_n\,(x)\,z^n=\frac{z+iz^2}{1-xz-z^2},\ \textit{with}\ GF_n\,(x)=(1+iz)\,h_{n-1}(a_1,[-a_2]).\tag{10}$$

Multiplying Equation (9) by $(x+2i+(2-ix)\,z)$, we obtain

$$\sum_{n=0}^{+\infty}(x+2i+(2-ix)\,z)\,h_{n-1}(a_1,[-a_2])z^n=\frac{(x+2i)\,z+(2-ix)\,z^2}{1-xz-z^2},$$

and we have the following theorem.

Theorem 2. *For $n\in\mathbb{N}$, the generating function of Gauss Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty}GL_n\,(x)\,z^n=\frac{(x+2i)\,z+(2-ix)\,z^2}{1-xz-z^2},\ \textit{with}\ GL_n\,(x)=(x+2i+(2-ix)\,z)\,h_{n-1}(a_1,[-a_2]).\tag{11}$$

By substituting $\begin{cases}a_1-a_2=2xy\\ a_1a_2=y\end{cases}$ in (6) and (7), we obtain

$$\sum_{n=0}^{+\infty}h_n(a_1,[-a_2])z^n=\frac{1}{1-2xyz-yz^2},\tag{12}$$

$$\sum_{n=0}^{+\infty}h_{n-1}(a_1,[-a_2])z^n=\frac{z}{1-2xyz-yz^2},\tag{13}$$

respectively, and we have the following theorem.

Theorem 3. *For $n\in\mathbb{N}$, the generating function of bivariate Pell polynomials is given by*

$$\sum_{n=0}^{+\infty}P_n\,(x,y)\,z^n=\frac{z}{1-2xyz-yz^2},\ \textit{with}\ P_n\,(x,y)=h_{n-1}(a_1,[-a_2]).\tag{14}$$

Put $y=1$ and $x=y=1$ in the relation (14), we can state the following corollaries.

Corollary 1. *[12] For $n\in\mathbb{N}$, the generating function of Pell polynomials is given by*

$$\sum_{n=0}^{+\infty}P_n\,(x)\,z^n=\frac{z}{1-2xz-z^2},\ \textit{with}\ P_n\,(x)=h_{n-1}(a_1,[-a_2]).$$

Corollary 2. *[7] For $n\in\mathbb{N}$, the generating function of Pell numbers is given by*

$$\sum_{n=0}^{+\infty}P_nz^n=\frac{z}{1-2z-z^2},\ \textit{with}\ P_n=h_{n-1}(a_1,[-a_2]).$$

Multiplying Equation (12) by 2 and (13) by $2xy$, we obtain

$$\sum_{n=0}^{+\infty}(2h_n(a_1,[-a_2])-2xyh_{n-1}(a_1,[-a_2]))z^n=\frac{2-2xyz}{1-2xyz-yz^2},$$

and we have the following theorem.

Theorem 4. *For $n \in \mathbb{N}$, the generating function of bivariate Pell Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} Q_n(x,y) z^n = \frac{2-2xyz}{1-2xyz-yz^2}, \text{ with } Q_n(x,y) = 2h_n(a_1,[-a_2]) - 2xyh_{n-1}(a_1,[-a_2]). \quad (15)$$

Put $y=1$ and $x=y=1$ in the relation (15), we have following corollaries:

Corollary 3. *[12] For $n \in \mathbb{N}$, the generating function of Pell Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} Q_n(x) z^n = \frac{2-2xz}{1-2xz-z^2}, \text{ with } Q_n(x) = 2h_n(a_1,[-a_2]) - 2xh_{n-1}(a_1,[-a_2]).$$

Corollary 4. *[7] For $n \in \mathbb{N}$, the generating function of Pell Lucas numbers is given by*

$$\sum_{n=0}^{+\infty} Q_n z^n = \frac{2-2z}{1-2z-z^2}, \text{ with } Q_n = 2h_n(a_1,[-a_2]) - 2h_{n-1}(a_1,[-a_2]).$$

Choosing a_1 and a_2 such that $\begin{cases} a_1 - a_2 = x \\ a_1 a_2 = y \end{cases}$ and substituting in (6) and (7), we obtain

$$\sum_{n=0}^{+\infty} h_n(a_1,[-a_2]) z^n = \frac{1}{1-xz-yz^2}, \quad (16)$$

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1,[-a_2]) z^n = \frac{z}{1-xz-yz^2}, \quad (17)$$

respectively, thus we get the following theorem.

Theorem 5. *For $n \in \mathbb{N}$, the generating function of bivariate Fibonacci polynomials is given by*

$$\sum_{n=0}^{+\infty} F_n(x,y) z^n = \frac{z}{1-xz-yz^2}, \text{ with } F_n(x,y) = h_{n-1}(a_1,[-a_2]). \quad (18)$$

Put $y=1$ and $x=y=1$ in the relation (18), we get following corollaries:

Corollary 5. *[12] For $n \in \mathbb{N}$, the generating function of Fibonacci polynomials is given by*

$$\sum_{n=0}^{+\infty} F_n(x) z^n = \frac{z}{1-xz-z^2}, \text{ with } F_n(x) = h_{n-1}(a_1,[-a_2]).$$

Corollary 6. *[9] For $n \in \mathbb{N}$, the generating function of Fibonacci numbers is given by*

$$\sum_{n=0}^{+\infty} F_n z^n = \frac{z}{1-z-z^2}, \text{ with } F_n = h_{n-1}(a_1,[-a_2]).$$

Multiplying Equation (16) by 2 and (17) by x, we obtain

$$\sum_{n=0}^{+\infty} \left(2h_n(a_1,[-a_2]) - xh_{n-1}(a_1,[-a_2])\right) z^n = \frac{2-xz}{1-xz-yz^2},$$

and we have the following theorem.

Theorem 6. *For $n \in \mathbb{N}$, the generating function of bivariate Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} L_n(x,y) z^n = \frac{2-xz}{1-xz-yz^2}, \text{ with } L_n(x,y) = 2h_n(a_1,[-a_2]) - xh_{n-1}(a_1,[-a_2]). \quad (19)$$

Put $y=1$ and $x=y=1$ in the relation (19), we get following corollaries:

Corollary 7. *[6] For $n \in \mathbb{N}$, the generating function of Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} L_n(x) z^n = \frac{2 - xz}{1 - xz - z^2}, \text{ with } L_n(x) = 2h_n(a_1, [-a_2]) - xh_{n-1}(a_1, [-a_2]).$$

Corollary 8. *[13] For $n \in \mathbb{N}$, the generating function of Lucas numbers is given by*

$$\sum_{n=0}^{+\infty} L_n z^n = \frac{2 - z}{1 - z - z^2}, \text{ with } L_n = 2h_n(a_1, [-a_2]) - h_{n-1}(a_1, [-a_2]).$$

By substituting $\begin{cases} a_1 - a_2 = xy \\ a_1 a_2 = 2y \end{cases}$ in (6) and (7), we have

$$\sum_{n=0}^{+\infty} h_n(a_1, [-a_2]) z^n = \frac{1}{1 - xyz - 2yz^2}, \tag{20}$$

$$\sum_{n=0}^{+\infty} h_{n-1}(a_1, [-a_2]) z^n = \frac{z}{1 - xyz - 2yz^2}, \tag{21}$$

respectively. Hence we obtain the following theorem.

Theorem 7. *For $n \in \mathbb{N}$, the generating function of bivariate Jacobsthal polynomials is given by*

$$\sum_{n=0}^{+\infty} J_n(x, y) z^n = \frac{z}{1 - xyz - 2yz^2}, \text{ with } J_n(x, y) = h_{n-1}(a_1, [-a_2]). \tag{22}$$

Put $x = y = 1$ in the relation (22), we get the generating function of Jacobsthal numbers [13].

$$\sum_{n=0}^{+\infty} J_n z^n = \frac{z}{1 - z - 2z^2}, \text{with } J_n = h_{n-1}(a_1, [-a_2]).$$

Multiplying Equation (20) by 2 and (21) by xy, we obtain

$$\sum_{n=0}^{+\infty} (2h_n(a_1, [-a_2]) - xyh_{n-1}(a_1, [-a_2])) z^n = \frac{2 - xyz}{1 - xyz - 2yz^2},$$

and we have the following theorem.

Theorem 8. *For $n \in \mathbb{N}$, the generating function of bivariate Jacobsthal Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} j_n(x, y) z^n = \frac{2 - xyz}{1 - xyz - 2yz^2}, \text{ with } j_n(x, y) = 2h_n(a_1, [-a_2]) - xyh_{n-1}(a_1, [-a_2]). \tag{23}$$

Put $x = y = 1$ in the relationship (23) we get the generating function of Jacobsthal Lucas numbers [10].

$$\sum_{n=0}^{+\infty} j_n z^n = \frac{2 - z}{1 - z - 2z^2}, \text{ with } j_n = 2h_n(a_1, [-a_2]) - h_{n-1}(a_1, [-a_2]).$$

4. Conclusion

In this paper, by making use of Equation (6), we have derived some new generating functions for the Gauss Fibonacci polynomials, Gauss Lucas polynomials, bivariate Pell polynomials, bivariate Pell Lucas polynomials, bivariate Fibonacci polynomials, bivariate Lucas polynomials, bivariate Jacobsthal polynomials and bivariate Jacobsthal Lucas polynomials. The derived theorems and corollaries are based on symmetric functions and these polynomials.

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