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Testing the Robustness of Linear Programming* Using a Diet Problem on a Multi-Shop System

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

Time, raw materials and labour are some of the finite resources in the world. Due to this, Linear Programming* (LP) is adopted by key decision-markers as an innovative tool to wisely consume these resources. This paper test the strength of linear programming models and presents an optimal solution to a diet problem on a multi-shop system formulated as linear, integer linear and mixed-integer linear programming models. All three models gave different least optimal values, that is, in linear programming, the optimal cost was GHS15.26 with decision variables being continuous ($\mathbb{R}^+$) and discrete ($\mathbb{Z}^+$). The cost increased to GHS17.50 when the models were formulated as mixed-integer linear programming with decision variables also being continuous ($\mathbb{R}^+$) and discrete ($\mathbb{Z}^+$) and lastly GHS17.70 for integer linear programming with discrete ($\mathbb{Z}^+$) decision variables. The difference in optimal cost for the same problem under different search spaces sufficiently establish that, in programming, the search space undoubtedly affect the optimal value. Applications to most problems like the diet and scheduling problems periodically require both discrete and continuous decision variables. This makes integer and mixed-integer linear programming models also an effective way of solving most problems. Therefore, Linear Programming* is applicable to numerous problems due to its ability to provide different required solutions.

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1 Introduction

Linear Programming* is popularly used by key decision-makers as an innovative tool when dealing with limited resources. Since the dawn of operation research, the practical application of Linear Programming* (LP) has extended significantly [1]. Linear programming (lp) have it search space to be continuous ($\mathbb{R}^+$). Sometimes, decision makers look for specific results from different search spaces (discrete, continuous or both). If the required solution is discrete ($\mathbb{Z}^+$), then the decision maker will be dealing with integer linear programming (ilp). In other cases, the results could be discrete and continuous which falls under mixed-integer linear programming (MILP). Fig. 1 illustrate the link between these three programming models under Linear Programming*. In this paper, Linear Programming* (LP) will be used to classify the set of [linear programming (lp), integer linear programming (ilp) and mixed-integer linear programming (milp)] models.

Under each specified category, different techniques such as the general and dual simplex method, interior point method and the branch-and-bound method exist for computing their models. With the help of a computer software, inbuilt solvers such as CPLEX, Gurobi, and MINOS can be used to compute complex and large-scale problems.

According to Mangasarian and Wolberg [2], linear programming during the 90's was used to diagnose breast cancer at the University of Wisconsin Hospital in Madison. Out of the 166 case processed, the use of linear programming was able to successfully diagnose 165 patients. This is an empirical evidence that dates the application of linear programming.

Gilmore and Gomory addressed various issues faced by the paper industry such as how to balance the number of machines available when order are filled up. Their work suggested that, linear programming can help rectified this difficulties especially when there exist limited resources [3].

The application of linear programming in economics has been in existence over the past few decades. Linear programming have been compared to the econometric approach to measure cost efficiency in banking [4]. According to Samuelson [5], economist previously used marginal equalization as a rule for defining equilibrium but with the emergent of linear programming they realized a fault with their assumption. In electric power transmission network, linear programming plays a very important role in minimizing the circuit miles [6].

Dantzig over 50 years ago first solved linear programming problems using the simplex method he developed. As reported by Gonzaga [7], the simplex method was and still the most widely used algorithm for solving simple linear programming problems. The interior point method used to solve both linear and non-linear programming is mostly done by eliminating the inequality constraints and incorporating them to a logarithmic barrier function. This method has shown superior computational performance when used to solve linear and quadratic programming [8].

In this paper, a diet problem is formulated as a linear programming problem and solved using a multiple shop system. The formulated linear models are reformulated as integer and mixed-integer linear programming models for comparison based on objective value. There will also be discussion on the practical applicability of linear programming models. This paper is structured in five sections. Section 2 gives a brief description of linear programming, integer linear programming and mixed-integer linear programming. The models are organized in section 3 and the last two sections inhabits our results and conclusion respectively.

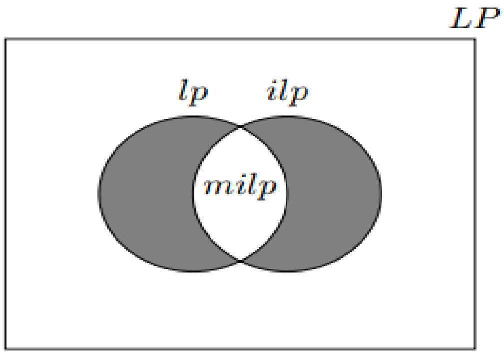

Fig. 1. A Venn Diagram Illustration of the link between LP, lp, ilp and milp

2 Materials and Methods

2.1 Linear programming

Linear programming (lp) has been used extensively throughout the years with its applications in economics, health, engineering, supply chain and many more [9]. The key concept of linear programming finds applications in almost all aspect of life such as; personal diet planning to maximize certain nutrient [10], minimization of transportation cost of goods from warehouse to retail customers [11]. Linear programming is branch of Operation Research that seeks to select the best method and deals with the application of scientific models in decision making. One important factor of linear programming is the size of the problem which is essentially the number of unknown variables and constraints. This factor measures the complexity of the problem [12].

Equation (2.1) is used as a point of reference in instantly comprehending what is meant by a linear function.

$$y(x) = Ax + b \tag{2.1}$$

where, $y(x)$ is the dependent variable, x is the independent variable, A is the gradient and b is a constant (Intercept when $x = 0$).

A linear function is a polynomial function where the degree of the independent variable is zero or one and there exists no cross-multiplication between the independent variables [13]. A similar concept is applied to linear programming but in this case, the objective function and the constraints assume the linear format [9]. In linear programming the search space is the set of all likely results that satisfies the objective function as well as all the constraints. The search space for an LP problem is a convex set. Equation (2.2) represent the canonical form of an LP.

$$\begin{aligned} \text{minimize/maximize} \quad & c^T x \\ \text{Subject to} \quad & Ax \leq b \\ & x_j \geq 0, x_j \in \mathbb{R}^+ \end{aligned} \tag{2.2}$$

In linear programming, there are case where multi-objectives exist [14]. Sometimes, accurate decision values can be attained by simultaneously solving multiple objective functions.

A shop attendant can decide to minimize the time spent in serving a customer and at the same time maximize the number of product purchased by the customer. A delivery company can minimize transportation cost and maximize delivery speed simultaneously.

Linear programming offers various domain of solution according to the specification of the decision marker. If a solution satisfies all the constraints of the LP, then the solution is said to be feasible. An optimal solution is a feasible solution where the objective function has reached its minimum or maximum value and there is no other feasible solution to either minimize or maximize the objective function [15].

2.2 Integer linear programming

Integer programming has it feasible or search space from the set of integers [7]. Integer programming can generally be written as:

$$\begin{aligned} &\text{minimize/maximize} && c^T x \\ &\text{subject to} && Ax \geq b, Bx \geq d, \\ & && x_j \geq 0\ , x_j \in \mathbb{Z} \end{aligned} \tag{2.3}$$

where the b and d are the right hand vectors, c the coefficient vector of the decision variables, and A and B are the matrices of the constraints [16].

A popular method for solving an integer linear programming problems is the branch and bound method proposed by LAND and DOIG with it's first appearance in the work of Little et al. [17]. The *Branch* and *bound* is a term originated from performing two operations (Branching and Bounding) [18].

In ILP, some problems are difficult to solve with the brute-force method due to the nature of some of their constraints. Lagrangian Relaxation (LR) can be used to approximate such a difficult problem by relaxing the constraint [19]. András Prékopa in 1979 worked on the first computer code in using the Lagrangian method to solve a scheduling problem [20]. Again, Wang and other colleagues in 1995 published a detailed article on the short-term power generation scheduling using an augmented Lagrangian Relaxation with the aim of minimizing the power system operation cost [21].

2.3 Mixed-integer linear programming

A practical mixed-integer linear programming problem is a linear programming problem that has some of its decision variables to be integer values at the optimal solution while others can equally be non-integers [22]. Therefore, mixed-integer linear programming is also an integer linear programming with non-integer decision variables. In 2002 Richards et al. published an article titled "Spacecraft Trajectory Planning with Avoidance Constraints Using Mixed-Integer Linear Programming". This group from Massachusetts Institute of Technology (MIT) with the aim to avoid collision of spacecrafts introduced avoidance constraints forming a mixed-integer linear programming. Their models were solved using the CPLEX solver in AMPL [23]. Mixed-integer linear programming follows the pattern of Integer Programming in Equation (2.3) above but not all decision variables will take integer solution at the optimal level.

$$\begin{aligned} &\text{minimize/maximize} && c^T x \\ &\text{subject to} && Ax \geq b, Bx \geq d, \\ & && x_{i,j} \geq 0\ , x_j \in \mathbb{R}\ , x_i \in \mathbb{Z} \end{aligned} \tag{2.4}$$

3 Models Formulation

3.1 Practical scenario

Consider a pregnant woman who has to visit the supermarket before heading to the clinic. On her previous visit, the doctor recommended to her a new diet for a healthy foetus development. In this scenario, the decision-maker (pregnant woman) wants to

1. minimize the cost of the items bought from the market.
2. satisfy each nutrients requirement based on the new diet plan.

It will be assumed that the supermarkets are in the same vicinity. Sufahani similarly used linear and integer programming to solve the diet planning for eczema patient drawing a successful conclusion that linear and integer programming approach will produce an optimal and feasible solution [24].

3.2 Nomenclature

Sets:

- S = Supermarkets.
- F = Food products.
- N = Nutrients

Supermarkets:
Three supermarkets will be used in this scenario (Supermarket 1, Supermarket 2, Supermarket 3).

Food Items:
Table 1 contains all the food items disposable to the pregnant woman at each supermarket.

Table 1. List of available food items

Egg	Milk	Chicken	Banana
Orange	Tomatoes	Wheat	Salmon
Carrot	Sardine	Yoghurt	Seal

Nutrients:
According to The American College of Obstetricians and Gynaecologists (ACOG), Table 2 are the key vitamins and minerals needed during pregnancy [25].

Table 2. List of essential nutrients for foetus development

Parameter	Description	Parameter	Description
$vitA$	vitamin A required	$vitC$	vitamin C required
$vitD$	vitamin D required	$iron$	Iron required
cal	Calcium required	$B12$	vitamin B12 required

Model parameters:
X_{Cost} = represent the cost of each food product. This cost will be indexed on the supermarkets and food items.

$X_{Nutrient}$ = represents the amount of nutrients available from each food item. This parameter is indexed on food and nutrient.

X_{Recom} = amount of recommended nutrients indexed on the nutrient set.

X_{Max} = the maximum amount of daily consumable food item.

Model variables:
X_{Buy} = decision variables indexed on the supermarket and food item.

3.3 Optimization models

The optimization models are formulated as a linear programming models.

3.3.1 Objective function

The objective is to minimize the cost of food purchased.

minimize Cost: $\sum_{s \in S, f \in F} X_{Cost}^{s,f} * X_{Buy}^{s,f}$

3.3.2 Constraints

In order to meet the doctor's diet recommendation, it is good that the required nutrients are satisfied irrespective of the cost involved. $X_{Nutrient}$ is the amount of nutrients each food product can provide as seen from Table 4.

In order to meet the recommended nutrients, the sum of each nutrient provided by each food purchased should be greater or equal to the required nutrients (X_{Recom}) listed in Table 3.

Subject to Vit A:
Vit A: $\sum_{s \in S, f \in F} X_{Nutrient}^{f,VitA} * X_{Buy}^{s,f} \geq X_{Recom}^{VitA}$

Subject to Vit C:
Vit C: $\sum_{s \in S, f \in F} X_{Nutrient}^{f,VitC} * X_{Buy}^{s,f} \geq X_{Recom}^{VitC}$

Subject to Vit A:
Vit D: $\sum_{s \in S, f \in F} X_{Nutrient}^{f,VitA} * X_{Buy}^{s,f} \geq X_{Recom}^{VitD}$

Subject to Vit A:
Iron: $\sum_{s \in S, f \in F} X_{Nutrient}^{f,Iron} * X_{Buy}^{s,f} \geq X_{Recom}^{Iron}$

Subject to Vit A:
Cal: $\sum_{s \in S, f \in F} X_{Nutrient}^{f,Cal} * X_{Buy}^{s,f} \geq X_{Recom}^{Cal}$

Subject to Vit A:
B12: $\sum_{s \in S, f \in F} X_{Nutrient}^{f,B12} * X_{Buy}^{s,f} \geq X_{Recom}^{B12}$

The above constraints can be put together as a single constraint;
Subject to Nutrient:

$\forall n \in$ NUTRIENT: $\sum_{s \in S, f \in F} X_{Nutrient}^{f,n} * X_{Buy}^{s,f} \geq X_{Recom}^{n}$

The following constraints are the boundary constraints on each food product.

subject to Egg:
Egg: $\sum_{s \in S} X_{Buy}^{s,Egg} \leq X_{Max}^{Egg}$

subject to Milk:
Milk: $\sum_{s \in S} X_{Buy}^{s,Milk} \leq X_{Max}^{Milk}$

subject to Chicken:
Chicken: $\sum_{s \in S} X_{Buy}^{s,Chicken} \leq X_{Max}^{Chicken}$

subject to Banana:
Banana: $\sum_{s \in S} X_{Buy}^{s,Banana} \leq X_{Max}^{Banana}$

subject to Orange:
Orange: $\sum_{s \in S} X_{Buy}^{s,Orange} \leq X_{Max}^{Orange}$

subject to Tomato:
Tomato: $\sum_{s \in S} X_{Buy}^{s,Tomato} \leq X_{Max}^{Tomato}$

subject to Wheat:
Wheat: $\sum_{s \in S} X_{Buy}^{s,Wheat} \leq X_{Max}^{Wheat}$

subject to Salmon:
Salmon: $\sum_{s \in S} X_{Buy}^{s,Salmon} \leq X_{Max}^{Salmon}$

subject to Carrot:
Carrot: $\sum_{s \in S} X_{Buy}^{s,Carrot} \leq X_{Max}^{Carrot}$

subject to Sardine:
Sardine: $\sum_{s \in S} X_{Buy}^{s,Sardine} \leq X_{Max}^{Sardine}$

subject to Yoghurt:
Yoghurt: $\sum_{s \in S} X_{Buy}^{s,Yoghurt} \leq X_{Max}^{Yoghurt}$

subject to Seal:
Seal: $\sum_{s \in S} X_{Buy}^{s,Seal} \leq X_{Max}^{Seal}$

Putting together the limits on each food product will yield;

subject to Limit:
$\forall f \in$ FOOD: $\sum_{s \in S} X_{Buy}^{s,f} \leq X_{Max}^{f}$

Subject to Non-negativity:
$\forall f \in$ FOOD, $s \in$ SHOP: $X_{Buy}^{s,f} \geq 0$

3.4 Optimization data

Table 3. The amount of daily recommended nutrients

$vitA$	770 RAE
$vitC$	85 mg
$vitD$	15 mcg
$iron$	27 mg
cal	1000 mg
$B12$	2.6 mcg

Table 4. The amount of each nutrient available from each food

	cal(mg)	$iron$ (mg)	$vitA$ (RAE)	$vitC$ (mg)	$vitD$ (mcg)	$B12$(mcg)
Egg(49.6g)	25	0.6	85	0	0.7	0.56
$Milk$(200ml)	369	0.3	104	0	2.9	2.8
$Chicken$(75g)	0	0.4	20	0	0.2	0.24
$Banana$(119g)	6	0.3	8	10	0	0
$Orange$(131g)	52	0.1	8	70	0	0
$Tomato$(170.1g)	12	0.3	52	16	0	0
$Wheat$(125ml)	10	5.3	0	0	0	0
$Salmon$(75g)	5	0.4	47	0	17	4.35
$Carrot$(61g)	20	0.2	367	4	0	0
$Sardine$(106g)	405	3.1	34	0	2.5	9.48
$Yoghurt$(200mL)	191	0.2	0	0	0.2	0.58
$Seal$(75g)	0	17.6	11	0	0	0

Table 5. The cost of food product (Ghana Cedis) at various supermarkets (SM) and allowable consumable food item

X_{Cost}	SM 1	SM 2	SM 3	X_{Max}
Egg(49.6g)	0.5	0.5	0.6	1
$Milk$(200ml)	3	3.5	3.2	1
$Chicken$(75g)	2	2	2	2
$Banana$(119g))	0.5	0.5	0.5	4
$Orange$(131g)	0.5	0.5	0.5	2
$Tomatoes$(170.1g)	0.4	0.45	0.45	2
$Wheat$(125ml)	2	1.5	1.5	1
$Salmon$(75g)	2	2	2	2
$Carrot$(61g)	1	1	1	2
$Sardine$(106g)	3.5	4	3.8	1
$Yogurt$(200mL)	1.8	1.8	1.8	1
$Seal$(75g)	3	3.5	3.5	1

4 Results and Discussion

4.1 Linear programming (lp)

The models were computed in AMPL using the CPLEX solver. Table 6 shows the optimization solution under the linear programming models.

The cost was evaluated to be GHS 15.26 computed in 5 dual simplex iterations. The computational time was 0.109375s. Using the solution space of $\mathbb{R}^+$ in linear programming problems the decisions variables are relaxed to assume decimals or whole values to achieve feasibility and optimality.

Table 6. The amount of food purchased at various supermarkets (SM)

X_{Buy}	SM1	SM2	SM3
Egg	0	1	0
Milk	1	0	0
Chicken	0	0	0
Banana	0	0	0
Orange	0	0	2
Tomatoes	0	0	0
Wheat	0	0.87936	0
Salmon	0.51992	0	0
Carrot	0	0	1.35031
Sardine	1	0	0
Yogurt	0	0	0.306809
Seal	1	0	0

4.2 Integer-linear programming (ilp)

In the linear programming problem above, the minimum cost was GHS 15.26, but since we can not have decision variables for amount of Salmon, Yogurt and Carrot to buy taking continuous values, we change the search space of all the decision variables is changed to integer solution space. Therefore,

Subject to Integer:
$\forall f \in$ FOOD, $s \in$ SHOP: $X_{Buy}^{f,s} \in \mathbb{Z}$

Table 7. The amount of food purchased at various supermarkets (SM)

X_{Buy}	SM1	SM2	SM3
Egg	0	0	0
Milk	1	0	0
Chicken	0	0	0
Banana	0	0	0
Orange	0	1	0
Tomatoes	1	0	0
Wheat	0	0	1
Salmon	0	0	1
Carrot	0	2	0
Sardine	1	0	0
Yogurt	0	0	1
Seal	1	0	0

In solving the linear models as an integer linear programming models, the solution got rid of all continuous values. The new cost of all food items purchased was GHS 17.70. 8 MIP simplex iterations were used within a computational time of 0.09375s. In integer linear programming, if an optimal solution rendering a minimum cost falls between two integer values such as 2 and 3, the integrality constraint discarded all decimals and decide on choosing 2 or 3.

4.3 Mixed-integer linear programming (milp)

Now let's assume that the supermarkets use scale and Wheat is actually measured in decimals.

Subject to Integer:
$\forall f \in$ FOOD, $s \in$ SHOP: $X_{Buy}^{f,s} \in \mathbb{Z}$
Subject to Non-Integer:
for wheat $\in$ FOOD, $s \in$ SHOP: $X_{Buy}^{Wheat,s} \in \mathbb{R}$
So relaxing one decision variable $X_{Buy}^{Wheat,s}$ reduces the objective function by placing it between linear programming and integer linear programming. The cost was GHS 17.50. 11 MIP simplex iterations were used within a computational time of 0.15625.

Table 8. The amount of food purchased at various supermarkets (SM)

X_{Buy}	SM1	SM2	SM3
Egg	0	0	0
Milk	1	0	0
Chicken	0	0	0
Banana	0	0	0
Orange	0	1	0
Tomatoes	1	0	0
Wheat	0	0.867925	0
Salmon	0	0	1
Carrot	0	2	0
Sardine	1	0	0
Yogurt	0	0	1
Seal	1	0	0

In a review work by Vielma, it was concluded that mixed-integer linear programming result in a more capable and/or smaller formulation for a wide class of problems [26]. Dooren in another review established that linear programming remain an essential tool for environmental optimization which exhibits considerable potential for finding solutions to varieties of diet problems [27].

5 Summary and Conclusion

A summary of the objective values, computational time and number of iterations exhibited by LP, ILP and MILP is summarize in Table 9.

Table 9. Comparison between LP, ILP, MILP

Criteria	LP	ILP	MILP
Objective Value (GHS)	15.26	17.7	17.5
Algorithm	Dual Simplex	MIP Simplex	MIP Simplex
Number of Iterations	5	8	11

Now comparing linear programming, integer linear programming and mixed-integer linear programming in terms of objective value, it was realized that linear programming offered the least cost due to the fact that in searching for minimum optimal solution, it has less restriction on the search space. Again, mixed-integer linear programming exhibited a cost above the linear programming but below integer linear programming. The reason for this is similar to the explanation given above for linear programming. In mixed-integer linear programming not all but some of the decision variable ($X_{Buy}^{Wheat,s}$) was relaxed while others were restricted to integer solution. These relaxed variables helped to minimize the objective value while assuming decimal values. Lastly, integer linear programming had the highest objective function value due to the high restriction on all the decision variables. In view of this, linear programming problems will always render a least cost due to it flexible search space.

From the above results, it was seen that mixed-integer linear programming and integer linear programming had the highest number of iterations. This indicate that mixed-integer linear programming and integer linear programming involved more steps than linear programming. This was due to the integer restriction imposed on some of the variables. The reason being that, extra steps were involved in making the decision variables integer after reaching optimality.

5.1 Conclusion

In this paper, the strength of linear programming was experimented using a diet problem on a multi-shop system. To test that linear programming can provide different solutions required by key decision-makers, a diet problem was used. The problem was first formulated as a linear programming (lp) model and reformulated into an integer linear programming (ilp) and mixed-integer linear model programming (milp) for different solutions. The difference in optimal cost for the same problem under different search spaces establish that, in programming, the search space affect the optimal value. The result also suggested that, linear programming can provide different suitable solutions. This achievement indicates that, integer and mixed-integer linear programming models are also effective in solving most real life problems. In the considered diet problem, the pregnant woman will spend GHS 17.50 to meet the required nutrients. The cost is lower in the case of linear programming but the decision variables value for the linear programming in this case are not realistic. Therefore integer or mixed-integer linear programming will be the appropriate approach. Based on the conclusions by [24, 27, 26], it can be establish that since integer and mixed-integer linear programming are special solution approach or subset of LP, Linear Programming* is therefore applicable to numerous problems due to its robustness and unique ability to provide different required solutions.

Competing Interests

Authors have declared that no competing interests exist.

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