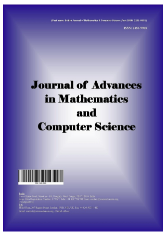

Journal of Advances in Mathematics and Computer Science

34(5): 1-13, 2019; Article no.JAMCS.53359

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

A New Generalization of Jacobsthal Lucas Numbers (Bi-Periodic Jacobsthal Lucas Sequence)

Sukran Uygun[1*] and Evans Owusu[1]

[1] *Department of Mathematics, Faculty of Science and Arts, Gaziantep University, Gaziantep, Turkey.*

Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2019/v34i530226

Editor(s):

(1) Prof. Dr. Sheng Zhang, Department of Mathematics, Bohai University, Jinzhou 121013, Jinzhou, China.

Reviewers:

(1) A. D. Godase, Dr. Babasaheb Ambedkar Marathwada University, India.

(2) Zhaolin Jiang, Linyi University, China.

Complete Peer review History: http://www.sdiarticle4.com/review-history/53359

Received: 18 October 2019

Accepted: 21 December 2019

Published: 02 January 2020

Original Research Article

Abstract

In this study, we bring into light a new generalization of the Jacobsthal Lucas numbers, which shall also be called the bi-periodic Jacobsthal Lucas sequence as

$$\hat{c}_n = \begin{cases} b\hat{c}_{n-1} + 2\hat{c}_{n-2}, & \text{if } n \text{ is even} \\ a\hat{c}_{n-1} + 2\hat{c}_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2,$$

with initial conditions $\hat{c}_0 = 2$, $\hat{c}_1 = a$. The Binet formula as well as the generating function for this sequence are given. The convergence property of the consecutive terms of this sequence is examined after which the well known Cassini, Catalan and the D'ocagne identities as well as some related summation formulas are also given.

Keywords: Bi-periodic Jacobsthal sequence; Jacobsthal Lucas sequence; Generalized Jacobsthal Lucas sequence; Generating function; Binet formula.

2010 Mathematics Subject Classification: 11B39, 11B83.

**Corresponding author: E-mail: suygun@gantep.edu.tr;*

1 Introduction

Due to the numerous applications of integer sequences such as Fibonacci, Lucas, Jacobsthal, Jacobsthal-Lucas, Pell etc in many fields of science and art, there have been many generalizations on them over the last century. In 1973, the first use of these numbers appears "A Handbook of Integer Sequences" in a paper by Sloane by the title applications of Jacobsthal sequences to curves [1]. You can see some of these different generalizations in our references [1], [2], [3]. In 1988, Horadam introduced the Jacobsthal and Jacobsthal–Lucas sequences recursively as where $n \geq 2$ as $j_n = j_{n-1} + 2j_{n-2}$,with the initial conditions $j_0 = 0,\ j_1 = 1$ and $c_n = c_{n-1} + 2c_{n-2}$, with the initial conditions $c_0 = 2,\ c_1 = 1$ respectively [4]. In [2], Horadam demonstrated the properties of the Jacobsthal and Jacobsthal–Lucas sequences in detail. Koshy's book [3] is an elaborate book for Fibonacci and Lucas numbers. For any natural number n and any nonzero real numbers a and b, the bi-periodic Fibonacci sequence, also known as the generalized Fibonacci sequence was defined recursively by Edson and Yayenie [5], [6] as

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd} \end{cases} \qquad n \geq 2$$

with initial conditions $q_0 = 0, q_1 = 1$. In the same way, the bi-periodic Lucas sequence was defined recursively by Bilgici [7] as

$$l_n = \begin{cases} bl_{n-1} + l_{n-2}, & \text{if } n \text{ is even} \\ al_{n-1} + l_{n-2}, & \text{if } n \text{ is odd} \end{cases} \qquad n \geq 2$$

with initial conditions $l_0 = 2$, $l_1 = a$. He also found some interesting identities between the above two sequences. The authors in [8], [9], [10], [11], [12], [13], [14], [15] gave interesting properties of bi-periodic sequences.

Fibonacci and Lucas sequences continued to receive a lot attention over the years. In [16], Uygun and Owusu defined a new generalization for the Jacobsthal sequence $\{\hat{\jmath}_n\}_{n=0}^{\infty}$, which they called the bi-periodic Jacobsthal sequence as

$$\hat{\jmath}_0 = 0,\ \hat{\jmath}_1 = 1,\ \hat{\jmath}_n = \begin{cases} a\hat{\jmath}_{n-1} + 2\hat{\jmath}_{n-2}, & \text{if } n \text{ is even} \\ b\hat{\jmath}_{n-1} + 2\hat{\jmath}_{n-2}, & \text{if } n \text{ is odd} \end{cases} \qquad n \geq 2.$$

They then obtained the Binet formula as follows

$$\hat{\jmath}_n = \frac{a^{1-\varepsilon(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right)$$

where $\lfloor a \rfloor$ is the floor function of a and $\varepsilon(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function. α and β are the roots of the nonlinear quadratic equation for the bi-periodic Jacobsthal sequence which is given as $x^2 - abx - 2ab = 0$. In [8], [9], [11] the authors carried bi-periodic sequences to bi-periodic Fibonacci, Lucas and Jacobsthal matrix sequences. The authors, in [12] gave interesting properties of bi-periodic Jacobsthal and bi-periodic Jacobsthal-Lucas sequences. Uygun and Karatas, in [17] introduced bi-periodic Pell-Lucas sequence. In [13], [14], Y. Choo examined some identities of generalized bi-periodic Fibonacci sequences. In [18], Gul studied on bi-periodic Jacobsthal and Jacobsthal-Lucas quaternions. In [15], Komatsu and Ramírez, gave convolutions of the bi-periodic Fibonacci numbers.

Now in this paper, just as the generalized Jacobsthal sequence and the others mentioned above, we define a new generalization for the Jacobsthal-Lucas sequence which we shall also call the bi-periodic Jacobsthal-Lucas sequence. We will then proceed to find its generating function as well as the Binet formula. The convergence properties of the consecutive terms of this sequence will be examined after which Cassini, Catalan and D'ocagne identities as well as some related formulas and properties will be given.

For any two non-zero real numbers a and b, the bi-periodic Jacobsthal-Lucas sequence denoted by $\{\hat{c}_n\}_{n=0}^{\infty}$ is defined recursively by

$$\hat{c}_0 = 2,\ \hat{c}_1 = a,\ \hat{c}_n = \begin{cases} b\hat{c}_{n-1} + 2\hat{c}_{n-2}, & \text{if } n \text{ is even} \\ a\hat{c}_{n-1} + 2\hat{c}_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad n \geq 2. \tag{1}$$

If $a = b = 1$ is chosen, we have the classic Jacobsthal-Lucas sequence. If we set $a = b = k$, where k can be any positive number, we get the $k-$Jacobsthal-Lucas sequence.

The first five elements of the bi-periodic Jacobsthal-Lucas sequence are

$$\hat{c}_0 = 2,\ \hat{c}_1 = a,\ \hat{c}_2 = ab + 4,\ \hat{c}_3 = a^2b + 6a,\quad \hat{c}_4 = a^2b^2 + 8ab + 8.$$

From (1), we obtain the nonlinear quadratic equation for the bi-periodic Jacobsthal-Lucas sequence as

$$x^2 - abx - 2ab = 0$$

with roots α and β defined by

$$\alpha = \frac{ab + \sqrt{a^2b^2 + 8ab}}{2},\quad \beta = \frac{ab - \sqrt{a^2b^2 + 8ab}}{2}. \tag{2}$$

2 Main Results

Lemma 1. *The bi-periodic Jacobsthal-Lucas sequence* $\{\hat{c}_n\}_{n=0}^{\infty}$ *satisfies the following properties:*

- $\hat{c}_{2n} = (ab + 4)\hat{c}_{2n-2} - 4\hat{c}_{2n-4},$
- $\hat{c}_{2n+1} = (ab + 4)\hat{c}_{2n-1} - 4\hat{c}_{2n-3}.$

Proof.

$$\begin{aligned} \hat{c}_{2n} &= b\hat{c}_{2n-1} + 2\hat{c}_{2n-2} \\ &= b(a\hat{c}_{2n-2} + 2\hat{c}_{2n-3}) + 2\hat{c}_{2n-2} \\ &= (ab + 2)\hat{c}_{2n-2} + 2(\hat{c}_{2n-2} - 2\hat{c}_{2n-4}) \\ &= (ab + 4)\hat{c}_{2n-2} - 4\hat{c}_{2n-4} \end{aligned}$$

The other proof can be done similarly. □

Lemma 2. α *and* β *defined by (2) satisfy the following properties;*

- $(\alpha + 2)(\beta + 2) = 4,$
- $\alpha + \beta = ab, \qquad \alpha\beta = -2ab,$
- $\beta + 2 = \frac{\beta^2}{ab}, \qquad \alpha + 2 = \frac{\alpha^2}{ab},$
- $-(\alpha + 2)\beta = 2\alpha, \qquad -(\beta + 2)\alpha = 2\beta.$

Proof. By using the given definitions of α and β, the identities above can easily to be proven. □

Theorem 2.1. *The generating function for the bi-periodic Jacobsthal-Lucas sequence is given by*

$$C(x)=\frac{2+ax-(ab+4)x^2+2ax^3}{1-(ab+4)x^2+4x^4} \tag{3}$$

Proof. The proof is done by using two different methods.

(1) The generating function for $C(x)$ can be represented in power series by

$$C(x)=\sum_{m=0}^{\infty}\hat{c}_m x^m=\hat{c}_0+\hat{c}_1x+....+\hat{c}_kx^k+...$$

By multiplying through this series by ax and $2x^2$ respectively and simplifying, we obtain

$$(1-ax-2x^2)C(x)=\hat{c}_0+x\hat{c}_1-ax\hat{c}_0+\sum_{m=2}^{\infty}(\hat{c}_m-a\hat{c}_{m-1}-2\hat{c}_{m-2})x^m.$$

By using the By using the Equation (1), we have

$$(1-ax-2x^2)C(x)=2+ax-2ax+\sum_{m=1}^{\infty}(\hat{c}_m-a\hat{c}_{m-1}-2\hat{c}_{m-2})x^{2m}.$$

Since $\hat{c}_{2m}=b\hat{c}_{2m-1}+2\hat{c}_{2m-2}$, we have

$$(1-ax-2x^2)C(x)=2-ax+\sum_{m=1}^{\infty}(b-a)\hat{c}_{2m-1}x^{2m}.$$

Now we define $c(x)$ as

$$c(x)=\sum_{m=1}^{\infty}\hat{c}_{2m-1}x^{2m-1}.$$

By using Lemma (1) and multiplying through $c(x)$ by $(ab+4)x^2$ and $4x^4$ respectively and simplifying as done above, we obtain

$$\begin{aligned}(1-(ab+4)x^2+4x^4)c(x) &= \sum_{m=1}^{\infty}\hat{c}_{2m-1}x^{2m-1}-(ab+4)\sum_{m=2}^{\infty}\hat{c}_{2m-3}x^{2m-1}\\ &\quad +4\sum_{m=3}^{\infty}\hat{c}_{2m-5}x^{2m-1}\\ &= (\hat{c}_1x+\hat{c}_3x^3)-(ab+4)\hat{c}_1x^3\\ &\quad +\sum_{m=3}^{\infty}(\hat{c}_{2m-1}-(ab+4)\hat{c}_{2m-3}+4\hat{c}_{2m-5})x^{2m-1}\\ &= ax+2ax^3+0.\end{aligned}$$

Hence

$$c(x)=\frac{ax+2ax^3}{1-(ab+4)x^2+4x^4}.$$

Plugging $c(x)$ into $C(x)$, we obtain

$$(1-ax-2x^2)C(x)=2-ax+(b-a)x\left(\frac{ax+2ax^3}{(1-(ab+4)x^2+4x^4)}\right)$$

By simplifying this equation, we get

$$C(x) = \frac{2 + ax - (ab+4)x^2 + 2ax^3}{1 - (ab+4)x^2 + 4x^4}.$$

which completes the proof.

(2) We want to show the other proof of this theorem.

$$C(x) = \sum_{m=0}^{\infty} \hat{c}_m x^m = \hat{c}_0(x) + \hat{c}_1(x) = \sum_{m=0}^{\infty} \hat{c}_{2m} x^{2m} + \sum_{m=0}^{\infty} \hat{c}_{2m+1} x^{2m+1}$$

We simplify the even part of the above series as follows

$$\hat{c}_0(x) = 2 + (ab+4)x^2 + \sum_{m=2}^{\infty} \hat{c}_{2m} x^{2m}$$

By multiplying through by $(ab+4)x^2$ and $4x^4$ respectively, we have

$$(ab+4)x^2 \hat{c}_0(x) = 2(ab+4)x^2 + (ab+4)\sum_{m=2}^{\infty} \hat{c}_{2m-2} x^{2m},$$

and

$$4x^4 \hat{c}_0(x) = 4\sum_{m=2}^{\infty} \hat{c}_{2m-4} x^{2m}.$$

By using Lemma (1), it is obtained that

$$\left[1 - (ab+4)x^2 + 4x^4\right] \hat{c}_0(x) = 2 - (ab+4)x^2.$$

Hence we get

$$\hat{c}_0(x) = \frac{2 - (ab+4)x^2}{1 - (ab+4)x^2 + 4x^4}.$$

Similarly, the odd part of the above series is simplified as follows

$$\hat{c}_1(x) = ax + (a^2b + 6a)x^3 + \sum_{m=2}^{\infty} \hat{c}_{2m+1} x^{2m+1}.$$

By multiplying through by $(ab+4)x^2$ and $4x^4$ respectively, we have

$$(ab+4)x^2 \hat{c}_1(x) = a(ab+4)x^3 + (ab+4)\sum_{m=2}^{\infty} \hat{c}_{2m-1} x^{2m+1},$$

and

$$4x^4 \hat{c}_1(x) = 4\sum_{m=2}^{\infty} \hat{c}_{2m-3} x^{2m+1}.$$

By using Lemma (1), it is obtained that

$$\left[1 - (ab+4)x^2 + 4x^4\right] \hat{c}_1(x) = ax + (a^2b + 6a)x^3 - a(ab+4)x^3 + 0.$$

Hence

$$\hat{c}_1(x) = \frac{ax + 2ax^3}{1 - (ab+4)x^2 + 4x^4}.$$

By adding the two results above, we obtain $C(x)$ as

$$C(x) = \frac{2 + ax - (ab+4)x^2 + 2ax^3}{1 - (ab+4)x^2 + 4x^4}.$$

which completes the proof. □

Theorem 2.2. *For every n belonging to the set of natural numbers, the Binet formula for the bi-periodic Jacobsthal-Lucas sequence is given by*

$$\hat{c}_n = \frac{a^{\varepsilon(n)}}{(ab)^{\left\lfloor \frac{n+1}{2} \right\rfloor}} \left(\alpha^n + \beta^n\right). \tag{4}$$

Proof. It must be noted that the parity function can also be expressed as

$$\varepsilon(n) = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$$

From the previous theorem, the generating function for the bi-periodic Jacobsthal-Lucas sequence can be written in partial fractions using partial fraction decomposition as,

$$C(x) = \frac{1}{4(\alpha-\beta)} \left[\frac{2a(\alpha+4)x - [2ab + (ab+4)\alpha]}{x^2 - \left(\frac{\alpha+2}{4}\right)} + \frac{-2a(\beta+4)x + [2ab + (ab+4)\beta]}{x^2 - \left(\frac{\beta+2}{4}\right)} \right].$$

The Maclaurin series expansion of the function $\frac{A-Bx}{x^2-C}$ is expressed in the form

$$\frac{A-Bx}{x^2-C} = \sum_{n=0}^{\infty} BC^{-n-1}x^{2n+1} - \sum_{n=0}^{\infty} AC^{-n-1}x^{2n}$$

Following the same order, the generating function $C(x)$ can be expanded as

$$\begin{aligned} C(x) &= \frac{1}{4(\alpha-\beta)} \left[\begin{array}{c} -\sum_{n=0}^{\infty} \left\{ 2a(\alpha+4)\left(\frac{\alpha+2}{4}\right)^{-n-1} \right\} x^{2n+1} \\ \sum_{n=0}^{\infty} [2ab + (ab+4)\alpha] \left(\frac{\alpha+2}{4}\right)^{-n-1} x^{2n} \end{array} \right] \\ &+ \frac{1}{4(\alpha-\beta)} \left[\begin{array}{c} \sum_{n=0}^{\infty} \left\{ 2a(\beta+4)\left(\frac{\beta+2}{4}\right)^{-n-1} \right\} x^{2n+1} \\ -\sum_{n=0}^{\infty} \left\{ [2ab + (ab+4)\beta] \left(\frac{\beta+2}{4}\right)^{-n-1} \right\} x^{2n} \end{array} \right], \end{aligned}$$

which can be expressed as

$$C(x) = \frac{1}{4(\alpha-\beta)} \left[\begin{array}{c} \sum_{n=0}^{\infty} \left\{ \begin{array}{c} -2a(\alpha+4)\left(\frac{4}{\alpha+2}\right)^{n+1} \\ 2a(\beta+4)\left(\frac{4}{\beta+2}\right)^{n+1} \end{array} \right\} x^{2n+1} \\ +\sum_{n=0}^{\infty} \left\{ \begin{array}{c} [2ab + (ab+4)\alpha]\left(\frac{4}{\alpha+2}\right)^{n+1} \\ -[2ab + (ab+4)\beta]\left(\frac{4}{\beta+2}\right)^{n+1} \end{array} \right\} x^{2n} \end{array} \right].$$

The above expression can be simplified as

$$\begin{aligned} C(x) &= \frac{1}{4(\alpha-\beta)} \left[\sum_{n=0}^{\infty} \left\{ 2a \left[\begin{array}{c} -(\alpha+4)(\beta+2)^{n+1} \\ +(\beta+4)(\alpha+2)^{n+1} \end{array} \right] \right\} x^{2n+1} \right] \\ &- \frac{1}{4(\alpha-\beta)} \left[\sum_{n=0}^{\infty} \left\{ \begin{array}{c} 2ab\left[-(\alpha+2)^{n+1} + (\beta+2)^{n+1}\right] \\ +(ab+4)\left[-\beta(\alpha+2)^{n+1} + \alpha(\beta+2)^{n+1}\right] \end{array} \right\} x^{2n} \right]. \end{aligned}$$

By using the identities in Lemma (2), we obtain

$$\begin{aligned} C(x) &= \frac{a}{(\alpha-\beta)} \left[\sum_{n=0}^{\infty} \left\{ \frac{1}{(ab)^{n+1}} \left[\alpha^{2n+1}(-ab+2\alpha) + \beta^{2n+1}(ab-2\beta)\right] \right\} x^{2n+1} \right] \\ &+ \frac{1}{4(\alpha-\beta)} \left[\sum_{n=0}^{\infty} \left\{ \frac{2}{(ab)^n} \left\{ \alpha^{2n}\left[(ab+4)\alpha - \alpha^2\right] - \beta^{2n}\left[(ab+4)\beta - \beta^2\right] \right\} \right\} x^{2n} \right]. \end{aligned}$$

Again using the Lemma (2), we have

$$C(x)=\sum_{n=0}^{\infty}\left\{\frac{a}{(ab)^{n+1}}\left(\alpha^{2n+1}+\beta^{2n+1}\right)\right\}x^{2n+1}+\sum_{n=0}^{\infty}\left\{\frac{1}{(ab)^{n}}(\alpha^{2n}+\beta^{2n})\right\}x^{2n}.$$

By the help of the parity function $\varepsilon(n)$, the above expansion can be condensed into the form

$$C(x)=\sum_{n=0}^{\infty}\frac{a^{\varepsilon(n)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}}\left(\alpha^{n}+\beta^{n}\right)x^{n}.$$

Hence by comparing the above with the generating function $C(x)=\sum_{n=0}^{\infty}C_nx^n$, we the desired result is obtained as

$$\hat{c}_n=\frac{a^{\epsilon(n)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}}\left(\alpha^{n}+\beta^{n}\right).$$

□

Theorem 2.3. *The limit of every two consecutive terms of the bi-periodic Jacobsthal-Lucas sequence is generalized as*

$$\lim_{n\to\infty}\frac{\hat{c}_{2n+1}}{\hat{c}_{2n}}=\frac{\alpha}{b},\qquad \lim_{n\to\infty}\frac{\hat{c}_{2n}}{\hat{c}_{2n-1}}=\frac{\alpha}{a}.$$

Proof. Taking into account that $|\beta|<\alpha$ and $\lim_{n\to\infty}\left(\frac{\beta}{\alpha}\right)^n=0$, we have

$$\lim_{n\to\infty}\frac{\hat{c}_{2n+1}}{\hat{c}_{2n}}=\lim_{n\to\infty}\frac{a/(ab)^{\lfloor\frac{2n+2}{2}\rfloor}}{1/(ab)^{\lfloor\frac{2n+1}{2}\rfloor}}\frac{\alpha^{2n+1}+\beta^{2n+1}}{\alpha^{2n}+\beta^{2n}}=\frac{a}{ab}\lim_{n\to\infty}\frac{1+\left(\frac{\beta}{\alpha}\right)^{2n+1}}{\frac{1}{\alpha}+\left(\frac{\beta}{\alpha}\right)^{2n+1}\frac{1}{\beta}}=\frac{\alpha}{b}.$$

The other proof can be done in a similar fashion. From this theorem we can conclude that the bi-periodic Jacobsthal-Lucas sequence does not converge. □

Theorem 2.4. *For any given integer n, we have*

$$\hat{c}_{-n}=(-2)^{-n}\hat{c}_n.$$

Proof. By using Binet's formula, it's obtained that

$$\begin{aligned}\hat{c}_{-n} &= \frac{a^{\varepsilon(-n)}}{(ab)^{\lfloor\frac{-n+1}{2}\rfloor}}\left(1/\alpha^{n}+1/\beta^{n}\right)=\frac{a^{\varepsilon(n)}}{(ab)^{\lfloor\frac{-n+1}{2}\rfloor}}\frac{\beta^n+\alpha^n}{(-2ab)^n}\\ &= \frac{a^{\varepsilon(n)}}{(ab)^{\lfloor\frac{n+1}{2}\rfloor}}\frac{\beta^n+\alpha^n}{(-2)^n}=(-2)^{-n}\hat{c}_n\end{aligned}$$

□

Theorem 2.5. *Let n be any nonnegative integer, then we have*

$$\sum_{k=0}^{n}\binom{n}{k}2^{n-k}a^{\varepsilon(k+1)}(ab)^{\lfloor\frac{k+1}{2}\rfloor}\hat{c}_k=a\hat{c}_{2n},$$

and

$$\sum_{k=0}^{n}\binom{n}{k}2^{n-k}a^{\varepsilon(k)}(ab)^{\lfloor\frac{k}{2}\rfloor}\hat{c}_{k+1}=\hat{c}_{2n+1}.$$

Proof. By using the Binet formula and Lemma (2), we get

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}2^{n-k}a^{\varepsilon(k+1)}(ab)^{\lfloor\frac{k+1}{2}\rfloor}\hat{c}_k \\
=& \sum_{k=0}^{n}\binom{n}{k}2^{n-k}a^{\varepsilon(k+1)}(ab)^{\lfloor\frac{k+1}{2}\rfloor}\frac{a^{\varepsilon(k)}}{(ab)^{\lfloor\frac{k+1}{2}\rfloor}}\left(\alpha^k+\beta^k\right) \\
=& a\sum_{k=0}^{n}\binom{n}{k}\alpha^k 2^{n-k}+a\sum_{p=0}^{n}\binom{n}{k}\beta^k 2^{n-k} \\
=& a(\alpha+2)^n+a(\beta+2)^n \\
=& a\left(\frac{\alpha^2}{ab}\right)^n+a\left(\frac{\beta^2}{ab}\right)^n=\hat{c}_{2n}
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}2^{n-k}a^{\varepsilon(k)}(ab)^{\lfloor\frac{k}{2}\rfloor}\hat{c}_{k+1} \\
=& \sum_{k=0}^{n}\binom{n}{k}2^{n-k}a^{\varepsilon(k)}(ab)^{\lfloor\frac{k}{2}\rfloor}\frac{a^{\epsilon(k+1)}}{(ab)^{\lfloor\frac{k+2}{2}\rfloor}}\left(\alpha^{k+1}+\beta^{k+1}\right) \\
=& \frac{\alpha}{b}\sum_{k=0}^{n}\binom{n}{k}\alpha^k 2^{n-k}-\frac{\beta}{b}\sum_{p=0}^{n}\binom{n}{k}\beta^k 2^{n-k} \\
=& \frac{\alpha}{b}(\alpha+2)^n+\frac{\beta}{b}(\beta+2)^n \\
=& \frac{a}{a}\left[\frac{\alpha}{b}\left(\frac{\alpha^2}{ab}\right)^n+\frac{\beta}{b}\left(\frac{\beta^2}{ab}\right)^n\right]=\hat{c}_{2n+1}
\end{aligned}$$

□

Theorem 2.6 (Catalan Identity)**.** *For all integers n and r, the Catalan Identity is given by*

$$\check{C}=\left(\frac{b}{a}\right)^{\varepsilon(n+r)}\hat{c}_{n-r}\hat{c}_{n+r}-\left(\frac{b}{a}\right)^{\varepsilon(n)}\hat{c}_n^2=\left(\frac{a}{b}\right)^{\varepsilon(r)}\frac{(\alpha-\beta)^2(-2)^{n-r}}{a^2}\hat{j}_r^2$$

Proof. By noting the identities below, the proof proceeds as follows;

$$\begin{aligned}
\varepsilon(n+r)+\left\lfloor\frac{n-r}{2}\right\rfloor+\left\lfloor\frac{n+r}{2}\right\rfloor &= n \\
\varepsilon(n+r)-\left\lfloor\frac{n-r+1}{2}\right\rfloor-\left\lfloor\frac{n+r+1}{2}\right\rfloor &= -n
\end{aligned}$$

$$\begin{aligned}
\left(\frac{b}{a}\right)^{\varepsilon(n+r)}\hat{c}_{n-r}\hat{c}_{n+r} &= \left(\frac{b}{a}\right)^{\varepsilon(n+r)}\left(\frac{\alpha^{n-r}+\beta^{n-r}}{a^{\lfloor\frac{n-r}{2}\rfloor}b^{\lfloor\frac{n-r+1}{2}\rfloor}}\right)\left(\frac{\alpha^{n+r}+\beta^{n+r}}{a^{\lfloor\frac{n+r}{2}\rfloor}b^{\lfloor\frac{n+r+1}{2}\rfloor}}\right) \\
&= \frac{b^{\varepsilon(n+r)-\lfloor\frac{n-r+1}{2}\rfloor-\lfloor\frac{n+r+1}{2}\rfloor}}{a^{\varepsilon(n+r)+\lfloor\frac{n-r}{2}\rfloor+\lfloor\frac{n+r}{2}\rfloor}}\left(\alpha^{n-r}+\beta^{n-r}\right)\left(\alpha^{n+r}+\beta^{n+r}\right) \\
&= (ab)^{-n}\left(\alpha^{2n}+\beta^{2n}+\alpha^{n-r}\beta^{n+r}+\alpha^{n+r}\beta^{n-r}\right).
\end{aligned}$$

Similarly,

$$\varepsilon(n)+2\left\lfloor\frac{n}{2}\right\rfloor=n,\qquad \varepsilon(n)-2\left\lfloor\frac{n+1}{2}\right\rfloor=-n,$$

$$\begin{aligned}\left(\frac{b}{a}\right)^{\varepsilon(n)}\hat{c}_n^2 &= \left(\frac{b}{a}\right)^{\varepsilon(n)}\frac{1}{a^{2\lfloor\frac{n}{2}\rfloor}b^{2\lfloor\frac{n+1}{2}\rfloor}}\left(\alpha^n+\beta^n\right)^2\\ &= \frac{b^{\varepsilon(n)-2\lfloor\frac{n+1}{2}\rfloor}}{a^{\varepsilon(n)+2\lfloor\frac{n}{2}\rfloor}}\left(\alpha^n+\beta^n\right)^2\\ &= (ab)^{-n}\left(\alpha^{2n}+\beta^{2n}+2\left(\alpha\beta\right)^n\right).\end{aligned}$$

$$\begin{aligned}\check{C} &= (ab)^{-n}\left[\alpha^{2n}+\beta^{2n}+\alpha^{n-r}\beta^{n+r}+\alpha^{n+r}\beta^{n-r}-\left(\alpha^{2n}+\beta^{2n}+2\left(\alpha\beta\right)^n\right)\right]\\ &= (ab)^{-n}\left[\alpha^{n-r}\beta^{n+r}+\alpha^{n+r}\beta^{n-r}-2\left(\alpha\beta\right)^n\right]\\ &= (ab)^{-n}\left(\alpha\beta\right)^n\left[\frac{\beta^r}{\alpha^r}+\frac{\alpha^r}{\beta^r}-2\right]\\ &= \frac{(-2ab)^n}{(ab)^n}\left[\frac{\beta^{2r}+\alpha^{2r}+2\alpha^r\beta^r}{(-2ab)^r}\right]\\ &= \frac{(-2)^{n-r}}{(ab)^r}\left(\beta^{2r}+\alpha^{2r}+2\alpha^r\beta\right)\\ &= \frac{(-2)^{n-r}}{(ab)^r}\left(\alpha^r-\beta^r\right)^2\end{aligned}$$

which completes the proof. □

Theorem 2.7 (**Cassini's Property or Simpson Property**). *For any number n belonging to the set of positive integers, we have*

$$\left(\frac{b}{a}\right)^{\varepsilon(n+1)}\hat{c}_{n-1}\hat{c}_{n+1}-\left(\frac{b}{a}\right)^{\varepsilon(n)}\hat{c}_n^2=(-2)^{n-1}\left(ab+8\right)$$

Proof. This is a special case of the Catalan Identity in which the value of r is 1. Therefore the Cassini' s Property can easily be proven by a mere substitution of $r=1$ into the Catalan Identity. □

Theorem 2.8 (**D'ocagne's Property**). *For any numbers m and n, belonging to the set of positive integers, with $m \geq n$, we have*

$$a^{\varepsilon(mn+m)}b^{\varepsilon(mn+n)}\hat{c}_{m+1}\hat{c}_n-a^{\varepsilon(mn+n)}b^{\varepsilon(mn+m)}\hat{c}_m\hat{c}_{n+1}=(-2)^n(ab+8)\hat{\jmath}_{m-n}.$$

Proof. By using the following equalities, we proceed as follows:

$$\varepsilon(m)+\varepsilon(n+1)-2\varepsilon(mn+m)=\varepsilon(m+1)+\varepsilon(n)-2\varepsilon(mn+n)=1-\varepsilon(m-n),\tag{5}$$

$$\varepsilon(m-n)=\varepsilon(mn+m)+\varepsilon(mn+n),\tag{6}$$

$$\frac{m+n-\varepsilon(m-n)}{2}=\left\lfloor\frac{m}{2}\right\rfloor+\left\lfloor\frac{n+1}{2}\right\rfloor-\varepsilon(mn+n),\tag{7}$$

$$\frac{m+n-\varepsilon(m-n)}{2} = \left\lfloor \frac{m+1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor - \varepsilon(mn+m). \tag{8}$$

By using the extended Binet's formula, (5), (6), (7), (8), it

$$\begin{aligned}
\psi &= a^{\varepsilon(mn+m)} b^{\varepsilon(mn+n)} \hat{c}_{m+1}\hat{c}_n \\
&= a^{\varepsilon(mn+m)} b^{\varepsilon(mn+n)} \frac{a^{1-\varepsilon(m)}}{(ab)^{\lfloor \frac{m}{2} \rfloor+1}} \frac{a^{1-\varepsilon(n+1)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left(\alpha^{m+1}+\beta^{m+1}\right)\left(\alpha^{n}+\beta^{n}\right) \\
&= \frac{ab^{\varepsilon(mn+n)} a^{1-\varepsilon(m)-\varepsilon(n+1)+\varepsilon(mn+m)}}{(ab)^{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n+1}{2} \rfloor+1}} (\alpha^{m+n+1}+\beta^{m+n+1}+\beta^{m+1}\alpha^{n}+\alpha^{m+1}\beta^{n}) \\
&= \frac{ab^{\varepsilon(mn+n)} a^{+\varepsilon(m-n)-\varepsilon(mn+m)}}{(ab)^{\frac{m+n-\varepsilon(m-n)}{2}+\varepsilon(mn+n)+1}} (\alpha^{m+n+1}+\beta^{m+n+1}+\beta^{m+1}\alpha^{n}+\alpha^{m+1}\beta^{n}) \\
&= \frac{a}{(ab)^{\frac{m+n-\varepsilon(m-n)}{2}+1}} [\alpha^{m+n+1}+\beta^{m+n+1}+\beta^{m+1}\alpha^{n}+\alpha^{m+1}\beta^{n}]
\end{aligned}$$

$$\begin{aligned}
\varphi &= a^{\varepsilon(mn+n)} b^{\varepsilon(mn+m)} \hat{c}_{m}\hat{c}_{n+1} \\
&= a^{\varepsilon(mn+n)} b^{\varepsilon(mn+m)} \frac{a^{1-\varepsilon(m+1)}}{(ab)^{\lfloor \frac{m+1}{2} \rfloor}} \frac{a^{1-\varepsilon(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor+1}} \left(\alpha^{n+1}+\beta^{n+1}\right)\left(\alpha^{m}+\beta^{m}\right) \\
&= \frac{ab^{\varepsilon(mn+m)} a^{1-\varepsilon(m+1)-\varepsilon(n)+\varepsilon(mn+n)}}{(ab)^{\lfloor \frac{m+1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor+1}} \left(\alpha^{n+1}+\beta^{n+1}\right)\left(\alpha^{m}+\beta^{m}\right) \\
&= \frac{ab^{\varepsilon(mn+m)} a^{+\varepsilon(m-n)-\varepsilon(mn+n)}}{(ab)^{\frac{m+n-\varepsilon(m-n)}{2}+\varepsilon(mn+m)+1}} \left[\alpha^{m+n+1}+\beta^{m+n+1}+\alpha^{m}\beta^{n+1}+\alpha^{m+1}\beta^{m}\right] \\
&= \frac{a}{(ab)^{\frac{m+n-\varepsilon(m-n)}{2}+1}} \left[\alpha^{m+n+1}+\beta^{m+n+1}+\beta^{m}\alpha^{n+1}+\alpha^{m}\beta^{n+1}\right]
\end{aligned}$$

From the above results, we obtain

$$\begin{aligned}
\psi-\varphi &= \left(\frac{a}{(ab)^{\frac{m+n-\varepsilon(m-n)}{2}+1}}\right) (\alpha^{m}\beta^{n})(\alpha-\beta) - (\alpha^{m}\beta^{n})(\alpha-\beta) \\
&= \left(\frac{a}{(ab)^{\frac{m+n-\varepsilon(m-n)}{2}+1}}\right) \frac{(\alpha-\beta)^2(\alpha\beta)^n\left[\alpha^{m-n}-\beta^{m-n}\right]}{\alpha-\beta} \\
&= \left(\frac{(-2)^n a}{(ab)^{\frac{m-n-\varepsilon(m-n)}{2}+1}}\right) \frac{(\alpha-\beta)^2\left[\alpha^{m-n}-\beta^{m-n}\right]}{\alpha-\beta} \\
&= \left(\frac{(-2)^n a}{(ab)^{\lfloor \frac{m-n}{2} \rfloor+1}}\right) (\alpha-\beta)^2 \frac{\alpha^{m-n}-\beta^{m-n}}{(\alpha-\beta)} \\
&= (-2)^n a^{\varepsilon(m-n)}(ab+8)\hat{\jmath}_{m-n}.
\end{aligned}$$

So, the proof is completed. □

Theorem 2.9. *Let $ab \neq 1$, then the sum of the first n elements of bi-periodic of Jacobsthal Lucas sequence is given as*

$$\sum_{k=0}^{n-1} \hat{c}_k = \frac{4\hat{c}_{n-2}+4\hat{c}_{n-1}-\hat{c}_n-\hat{c}_{n+1}-(ab+4)+2+3a}{1-ab}$$

Proof. Let n even. By using Binet formula for bi-periodic of Jacobsthal Lucas sequence, we get

$$\begin{aligned}\sum_{k=0}^{n-1}\hat{c}_k &= \sum_{k=0}^{\frac{n-2}{2}}\hat{c}_{2k}+\sum_{k=0}^{\frac{n-2}{2}}\hat{c}_{2k+1}\\ &= \sum_{k=0}^{\frac{n-2}{2}}\left\{\frac{1}{(ab)^k}\left(\alpha^{2k}+\beta^{2k}\right)+\frac{a}{(ab)^{k+1}}\left(\alpha^{2k+1}+\beta^{2k+1}\right)\right\}\end{aligned}$$

If we use the property of geometric series, we get

$$\begin{aligned}&= \left(\frac{\alpha^n-(ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}-1}(\alpha^2-ab)}+\frac{\beta^n-(ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}-1}(\beta^2-ab)}\right)\\ &\quad +a\left(\frac{\alpha^{n+1}-\alpha(ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}}(\alpha^2-ab)}+\frac{\beta^{n+1}-\beta(ab)^{\frac{n}{2}}}{(ab)^{\frac{n}{2}}(\beta^2-ab)}\right)\end{aligned}$$

After some algebraic operations we have

$$\begin{aligned}&= \frac{1}{(1-ab)(ab)^{\frac{n}{2}+1}}\left(\begin{array}{c}4a^2b^2(\alpha^{n-2}+\beta^{n-2})-ab(\alpha^n+\beta^n)\\ -(ab)^{\frac{n}{2}}(\alpha^2+\beta^2)+2(ab)^{\frac{n}{2}+1}\end{array}\right)\\ &\quad +\frac{a}{(1-ab)(ab)^{\frac{n}{2}+2}}\left(4a^2b^2(\alpha^{n-1}+\beta^{n-1})-ab(\alpha^{n+1}+\beta^{n+1})+3(ab)^{\frac{n}{2}+1}(\alpha+\beta)\right)\\ &= \frac{4\hat{c}_{n-2}+4\hat{c}_{n-1}-\hat{c}_n-\hat{c}_{n+1}-(ab+4)+2+3a}{1-ab}\end{aligned}$$

If we make the similar operation for the odd elements of bi-periodic Jacobsthal Lucas sequence we obtain

$$\begin{aligned}\sum_{k=0}^{n-1}\hat{c}_k &= \sum_{k=0}^{\frac{n-1}{2}}\hat{c}_{2k}+\sum_{k=0}^{\frac{n-3}{2}}\hat{c}_{2k+1}\\ &= \sum_{k=0}^{\frac{n-1}{2}}\frac{1}{(ab)^k}\left(\alpha^{2k}+\beta^{2k}\right)+\sum_{k=0}^{\frac{n-3}{2}}\frac{a}{(ab)^{k+1}}\left(\alpha^{2k+1}+\beta^{2k+1}\right)\end{aligned}$$

If we use the property of geometric series, we get

$$\begin{aligned}&= \left(\frac{\alpha^{n+1}-(ab)^{\frac{n+1}{2}}}{(ab)^{\frac{n-1}{2}}(\alpha^2-ab)}+\frac{\beta^{n+1}-(ab)^{\frac{n+1}{2}}}{(ab)^{\frac{n-1}{2}}(\beta^2-ab)}\right)\\ &\quad +\left(\frac{\alpha^n-\alpha(ab)^{\frac{n-1}{2}}}{(ab)^{\frac{n-1}{2}}(\alpha^2-ab)}+\frac{\beta^n-\beta(ab)^{\frac{n-1}{2}}}{(ab)^{\frac{n-1}{2}}(\beta^2-ab)}\right)\end{aligned}$$

After some algebraic operations we have

$$\begin{aligned}&= \frac{1}{(1-ab)(ab)^{\frac{n+3}{2}}}\left(\begin{array}{c}4a^2b^2(\alpha^{n-1}+\beta^{n-1})-ab(\alpha^{n+1}+\beta^{n+1})\\ -(ab)^{\frac{n+1}{2}}(\alpha^2+\beta^2)+2(ab)^{\frac{n+3}{2}}\end{array}\right)\\ &\quad +\frac{a}{(1-ab)(ab)^{\frac{n+3}{2}}}\left(4a^2b^2(\alpha^{n-2}+\beta^{n-2})-ab(\alpha^n+\beta^n)+3(ab)^{\frac{n+1}{2}}(\alpha+\beta)\right)\\ &= \frac{4\hat{c}_{n-2}+4\hat{c}_{n-1}-\hat{c}_n-\hat{c}_{n+1}-(ab+4)+2+3a}{1-ab}\end{aligned}$$

If we combine the results we get the desired result. □

3 Conclusions

In this study we define a new generalization of Jacobsthal Lucas sequence which is called bi-periodic Jacobsthal Lucas sequence and find basic properties of the sequences such as Binet formula, generating function, the sum of the first n elements, D'ocagne, Catalan, Cassini.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Sloane NJA. A handbook of integer sequences. New York Academic Press; 1973.

[2] Horadam AF. Jacobsthal representation numbers. The Fibonacci Quarterly. 1996;37(2):40-54.

[3] Koshy T. Fibonacci and Lucas numbers with applications. John Wiley and Sons Inc., NY; 2001.

[4] Horadam AF. Jacobsthal and Pell curves. The Fibonacci Quarterly. 1988;26.1:79-83.

[5] Edson M, Yayenie O. A new generalization of Fibonacci sequences and extended Binet's formula. Integers. 2009;9:639-654.

[6] Yayenie O. A Note on Generalized Fibonacci Sequence. Applied Mathematics and Computation. 2011;217:5603-5611.

[7] Bilgici G. Two generalizations of Lucas sequence. Applied Mathematics and Computation. 2014;245:526-538.

[8] Coskun A, Taskara N. The matrix sequence in terms of bi-periodic Fibonacci numbers. arXiv:1603.07487v2 [math.NT]. 4 Apr 2016.

[9] Coskun A, Yilmaz N, Taskara N. A note on the bi-periodic Fibonacci and Lucas matrix sequences. arXiv:1604.00766v1 [math.NT]. 4 Apr 2016.

[10] Jun SP, Choi KH. Some properties of the generalized Fibonacci sequence $\{q_n\}$ by matrix methods. The Korean Journal of Mathematics. 2016;24(4):681-691.

[11] Uygun S, Owusu E. Matrix representation of bi-periodic Jacobsthal. arXiv:1702.00604v1 [math.CO]. 2 Feb 2017.

[12] Uygun S, Karatas H, Akıncı E. Relations on bi-periodic Jacobsthal sequence. Transylvanian Journal of Mathematics and Mechanics. 2018;10(2):141-151.

[13] Choo Y. On the generalizations of Fibonacci identities. Results in Mathematics. 2017;71:347–356.

[14] Choo Y. Some identities on generalized bi-periodic Fibonacci sequences. International Journal of Mathematical Analysis. 2019;13(6):259-267.

[15] Komatsu T, Ramírez JL. Convolutions of the bi-periodic Fibonacci numbers. Hacettepe Journal of Mathematics & Statistics. 2019;Early Access:1–13.

[16] Uygun S, Owusu E. A new generalization of Jacobsthal numbers (Bi-Periodic Jacobsthal Sequences). Journal of Mathematical Analysis. 2016;7(5):28-39.

[17] Uygun S, Karatas H. A new generalization of Pell-Lucas numbers (Bi-Periodic Pell-Lucas Sequence). Communications in Mathematics and Applications. 2019;10(3):1–12.

[18] Gul K. On Bi-periodic Jacobsthal and Jacobsthal-Lucas Quaternions. Journal of Mathematics Research. 2019;11(2):44-52.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sdiarticle4.com/review-history/53359