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Properties of Generalized 6-primes Numbers

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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ABSTRACT

In this paper, we introduce the generalized 6-primes sequence and we deal with, in detail, three special cases which we call them 6-primes, Lucas 6-primes and modified 6-primes sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

Keywords: Hexanacci numbers; 6-primes numbers; Lucas 6-primes numbers; modified 6-primes numbers.

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1 INTRODUCTION

In this paper, we define the generalized 6-primes sequences and we investigate, in detail, three special cases which we call them 6-primes, Lucas-6-primes and modified 6-primes sequences.

The sequence of Fibonacci numbers $\{F_n\}$ and the sequence of Lucas numbers $\{L_n\}$ are defined by

$$F_n = F_{n-1} + F_{n-2}, \;\; n \geq 2, \quad F_0 = 0, \; F_1 = 1,$$

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and

$$L_n = L_{n-1} + L_{n-2}, \ \ n \geq 2, \quad L_0 = 2, \ L_1 = 1$$

respectively. The generalizations of Fibonacci and Lucas sequences produce several nice and interesting sequences.

The generalized Hexanacci sequence $\{W_n(W_0, W_1, W_2, W_3, W_4, W_5; r_1, r_2, r_3, r_4, r_5, r_6)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined by the sixth-order recurrence relation

$$\begin{aligned} W_n &= r_1 W_{n-1} + r_2 W_{n-2} + r_3 W_{n-3} + r_4 W_{n-4} + r_5 W_{n-5} + r_6 W_{n-6}, \\ W_0 &= c_0, W_1 = c_1, W_2 = c_2, \ W_3 = c_3, W_4 = c_4, W_5 = c_5, \ n \geq 6 \end{aligned} \tag{1.1}$$

where $W_0, W_1, W_2, W_3, W_4, W_5$ are arbitrary real or complex numbers and $r_1, r_2, r_3, r_4, r_5, r_6$ are real numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r_5}{r_6} W_{-n+1} - \frac{r_4}{r_6} W_{-n+2} - \frac{r_3}{r_6} W_{-n+3} - \frac{r_2}{r_6} W_{-n+4} - \frac{r_1}{r_6} W_{-n+5} + \frac{1}{r_6} W_{-n+6}$$

for $n = 1, 2, 3, ...$ when $r_6 \neq 0$. Therefore, recurrence (1.1) holds for all integers n. Hexanacci sequence has been studied by many authors, see for example [1], [2].

As $\{W_n\}$ is a sixth-order recurrence sequence (difference equation), it's characteristic equation is

$$x^6 - r_1 x^5 - r_2 x^4 - r_3 x^3 - r_4 x^2 - r_5 x - r_6 = 0 \tag{1.2}$$

whose roots are $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$. Generalized Hexanacci numbers can be expressed, for all integers n, using Binet's formula.

Theorem 1.1. *(Binet's formula of generalized Hexanacci numbers)*

$$W_n = \sum_{k=1}^{6} \frac{b_k \alpha_k^n}{\prod\limits_{\substack{j=1 \\ k\neq j}}^{6} (\alpha_k - \alpha_j)} \tag{1.3}$$

where

$$\begin{aligned}
b_1 &= W_5 - (\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6)W_4 + (\alpha_2\alpha_5+\alpha_2\alpha_3+\alpha_2\alpha_6+\alpha_5\alpha_3+\alpha_5\alpha_6+\alpha_2\alpha_4+\alpha_5\alpha_4+\alpha_3\alpha_6+\alpha_3\alpha_4+\alpha_6\alpha_4)W_3 \\
&\quad -(\alpha_2\alpha_5\alpha_3+\alpha_2\alpha_5\alpha_6+\alpha_2\alpha_5\alpha_4+\alpha_2\alpha_3\alpha_6+\alpha_5\alpha_3\alpha_6+\alpha_2\alpha_3\alpha_4+\alpha_2\alpha_6\alpha_4+\alpha_5\alpha_3\alpha_4+\alpha_5\alpha_6\alpha_4+\alpha_3\alpha_6\alpha_4)W_2 \\
&\quad +(\alpha_2\alpha_5\alpha_3\alpha_6+\alpha_2\alpha_5\alpha_3\alpha_4+\alpha_2\alpha_5\alpha_6\alpha_4+\alpha_2\alpha_3\alpha_6\alpha_4+\alpha_5\alpha_3\alpha_6\alpha_4)W_1 - \alpha_2\alpha_5\alpha_3\alpha_6\alpha_4 W_0, \\
b_2 &= W_5 - (\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6)W_4 + (\alpha_1\alpha_5+\alpha_1\alpha_3+\alpha_1\alpha_6+\alpha_1\alpha_4+\alpha_5\alpha_3+\alpha_5\alpha_6+\alpha_5\alpha_4+\alpha_3\alpha_6+\alpha_3\alpha_4+\alpha_6\alpha_4)W_3 \\
&\quad -(\alpha_1\alpha_5\alpha_3+\alpha_1\alpha_5\alpha_6+\alpha_1\alpha_5\alpha_4+\alpha_1\alpha_3\alpha_6+\alpha_1\alpha_3\alpha_4+\alpha_1\alpha_6\alpha_4+\alpha_5\alpha_3\alpha_6+\alpha_5\alpha_3\alpha_4+\alpha_5\alpha_6\alpha_4+\alpha_3\alpha_6\alpha_4)W_2 \\
&\quad +(\alpha_1\alpha_5\alpha_3\alpha_6+\alpha_1\alpha_5\alpha_3\alpha_4+\alpha_1\alpha_5\alpha_6\alpha_4+\alpha_1\alpha_3\alpha_6\alpha_4+\alpha_5\alpha_3\alpha_6\alpha_4)W_1 - \alpha_1\alpha_5\alpha_3\alpha_6\alpha_4 W_0, \\
b_3 &= W_5 - (\alpha_1+\alpha_2+\alpha_4+\alpha_5+\alpha_6)W_4 + (\alpha_1\alpha_2+\alpha_1\alpha_5+\alpha_1\alpha_6+\alpha_2\alpha_5+\alpha_1\alpha_4+\alpha_2\alpha_6+\alpha_5\alpha_6+\alpha_2\alpha_4+\alpha_5\alpha_4+\alpha_6\alpha_4)W_3 \\
&\quad -(\alpha_1\alpha_2\alpha_5+\alpha_1\alpha_2\alpha_6+\alpha_1\alpha_5\alpha_6+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_5\alpha_4+\alpha_2\alpha_5\alpha_6+\alpha_1\alpha_6\alpha_4+\alpha_2\alpha_5\alpha_4+\alpha_2\alpha_6\alpha_4+\alpha_5\alpha_6\alpha_4)W_2 \\
&\quad +(\alpha_1\alpha_2\alpha_5\alpha_6+\alpha_1\alpha_2\alpha_5\alpha_4+\alpha_1\alpha_2\alpha_6\alpha_4+\alpha_1\alpha_5\alpha_6\alpha_4+\alpha_2\alpha_5\alpha_6\alpha_4)W_1 - \alpha_1\alpha_2\alpha_5\alpha_6\alpha_4 W_0, \\
b_4 &= W_5 - (\alpha_1+\alpha_2+\alpha_3+\alpha_5+\alpha_6)W_4 + (\alpha_1\alpha_2+\alpha_1\alpha_5+\alpha_1\alpha_3+\alpha_1\alpha_6+\alpha_2\alpha_5+\alpha_2\alpha_3+\alpha_2\alpha_6+\alpha_5\alpha_3+\alpha_5\alpha_6+\alpha_3\alpha_6)W_3 \\
&\quad -(\alpha_1\alpha_2\alpha_5+\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_6+\alpha_1\alpha_5\alpha_3+\alpha_1\alpha_5\alpha_6+\alpha_1\alpha_3\alpha_6+\alpha_2\alpha_5\alpha_3+\alpha_2\alpha_5\alpha_6+\alpha_2\alpha_3\alpha_6+\alpha_5\alpha_3\alpha_6)W_2 \\
&\quad +(\alpha_1\alpha_2\alpha_5\alpha_3+\alpha_1\alpha_2\alpha_5\alpha_6+\alpha_1\alpha_2\alpha_3\alpha_6+\alpha_1\alpha_5\alpha_3\alpha_6+\alpha_2\alpha_5\alpha_3\alpha_6)W_1 - \alpha_1\alpha_2\alpha_5\alpha_3\alpha_6 W_0, \\
b_5 &= W_5 - (\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_6)W_4 + (\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_1\alpha_6+\alpha_1\alpha_4+\alpha_2\alpha_3+\alpha_2\alpha_6+\alpha_2\alpha_4+\alpha_3\alpha_6+\alpha_3\alpha_4+\alpha_6\alpha_4)W_3 \\
&\quad -(\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_6+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_3\alpha_6+\alpha_1\alpha_3\alpha_4+\alpha_1\alpha_6\alpha_4+\alpha_2\alpha_3\alpha_6+\alpha_2\alpha_3\alpha_4+\alpha_2\alpha_6\alpha_4+\alpha_3\alpha_6\alpha_4)W_2 \\
&\quad +(\alpha_1\alpha_2\alpha_3\alpha_6+\alpha_1\alpha_2\alpha_3\alpha_4+\alpha_1\alpha_2\alpha_6\alpha_4+\alpha_1\alpha_3\alpha_6\alpha_4+\alpha_2\alpha_3\alpha_6\alpha_4)W_1 - \alpha_1\alpha_2\alpha_3\alpha_6\alpha_4 W_0, \\
b_6 &= W_5 - (\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5)W_4 + (\alpha_1\alpha_2+\alpha_1\alpha_5+\alpha_1\alpha_3+\alpha_2\alpha_5+\alpha_1\alpha_4+\alpha_2\alpha_3+\alpha_5\alpha_3+\alpha_2\alpha_4+\alpha_5\alpha_4+\alpha_3\alpha_4)W_3 \\
&\quad -(\alpha_1\alpha_2\alpha_5+\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_5\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_5\alpha_4+\alpha_2\alpha_5\alpha_3+\alpha_1\alpha_3\alpha_4+\alpha_2\alpha_5\alpha_4+\alpha_2\alpha_3\alpha_4+\alpha_5\alpha_3\alpha_4)W_2 \\
&\quad +(\alpha_1\alpha_2\alpha_5\alpha_3+\alpha_1\alpha_2\alpha_5\alpha_4+\alpha_1\alpha_2\alpha_3\alpha_4+\alpha_1\alpha_5\alpha_3\alpha_4+\alpha_2\alpha_5\alpha_3\alpha_4)W_1 - \alpha_1\alpha_2\alpha_5\alpha_3\alpha_4 W_0.
\end{aligned}$$

Usually, it is customary to choose $r_1, r_2, r_3, r_4, r_5, r_6$ so that the Equ. (1.2) has at least one real (say α_1) solutions.

Note that the Binet form of a sequence satisfying (1.2) for non-negative integers is valid for all integers $n,$ for a proof of this result see [3]. This result of Howard and Saidak [3] is even true in the case of higher-order recurrence relations.

In this paper, we consider the case $r_1 = 2, r_2 = 3, r_3 = 5, r_4 = 7, r_5 = 11, r_6 = 13$ and in this case we write $V_n = W_n$. For recent relevant studies, see [4,5,6,7,8]. Note that $2, 3, 5, 7, 11$ and 13 are prime numbers. Prime numbers are numbers that have only 2 factors: 1 and themselves. For more details on prime numbers, see for example [9].

A generalized 6-primes sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2, V_3, V_4, V_5)\}_{n\geq 0}$ is defined by the sixth-order recurrence relations

$$V_n = 2V_{n-1} + 3V_{n-2} + 5V_{n-3} + 7V_{n-4} + 11V_{n-5} + 13V_{n-6} \tag{1.4}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2, V_3 = c_3, V_4 = c_4, V_5 = c_5$ not all being zero.

Note that the prime numbers $2, 3, 5, 7, 11$ and 13 are the coefficients of the difference equation (1.4). We investigate, in detail, the sequence which is given in (1.4). This is why the paper has the title 'Properties of Generalized 6-primes Numbers'.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{11}{13}V_{-(n-1)} - \frac{7}{13}V_{-(n-2)} - \frac{5}{13}V_{-(n-3)} - \frac{3}{13}V_{-(n-4)} - \frac{2}{13}V_{-(n-5)} + \frac{1}{13}V_{-(n-6)}$$

for $n = 1, 2, 3, ...$Therefore, recurrence (1.4) holds for all integer $n.$

(1.3) can be used to obtain Binet's formula of generalized 6-primes numbers. Binet's formula of generalized 6-primes numbers can be given as

$$V_n = \sum_{k=1}^{6} \frac{b_k \theta_k^n}{\prod\limits_{\substack{j=1\\ k\neq j}}^{6} (\theta_k - \theta_j)} \tag{1.5}$$

where

$$\begin{aligned}
b_1 \quad = \quad & V_5 - (\theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6)V_4 + (\theta_2\theta_5 + \theta_2\theta_3 + \theta_2\theta_6 + \theta_5\theta_3 + \theta_5\theta_6 + \theta_2\theta_4 + \theta_5\theta_4 + \theta_3\theta_6 + \theta_3\theta_4 + \theta_6\theta_4)V_3 \\
& -(\theta_2\theta_5\theta_3 + \theta_2\theta_5\theta_6 + \theta_2\theta_5\theta_4 + \theta_2\theta_3\theta_6 + \theta_5\theta_3\theta_6 + \theta_2\theta_3\theta_4 + \theta_2\theta_6\theta_4 + \theta_5\theta_3\theta_4 + \theta_5\theta_6\theta_4 + \theta_3\theta_6\theta_4)V_2 \\
& +(\theta_2\theta_5\theta_3\theta_6 + \theta_2\theta_5\theta_3\theta_4 + \theta_2\theta_5\theta_6\theta_4 + \theta_2\theta_3\theta_6\theta_4 + \theta_5\theta_3\theta_6\theta_4)V_1 - \theta_2\theta_5\theta_3\theta_6\theta_4V_0, \\
b_2 \quad = \quad & V_5 - (\theta_1 + \theta_3 + \theta_4 + \theta_5 + \theta_6)V_4 + (\theta_1\theta_5 + \theta_1\theta_3 + \theta_1\theta_6 + \theta_1\theta_4 + \theta_5\theta_3 + \theta_5\theta_6 + \theta_5\theta_4 + \theta_3\theta_6 + \theta_3\theta_4 + \theta_6\theta_4)V_3 \\
& -(\theta_1\theta_5\theta_3 + \theta_1\theta_5\theta_6 + \theta_1\theta_5\theta_4 + \theta_1\theta_3\theta_6 + \theta_1\theta_3\theta_4 + \theta_1\theta_6\theta_4 + \theta_5\theta_3\theta_6 + \theta_5\theta_3\theta_4 + \theta_5\theta_6\theta_4 + \theta_3\theta_6\theta_4)V_2 \\
& +(\theta_1\theta_5\theta_3\theta_6 + \theta_1\theta_5\theta_3\theta_4 + \theta_1\theta_5\theta_6\theta_4 + \theta_1\theta_3\theta_6\theta_4 + \theta_5\theta_3\theta_6\theta_4)V_1 - \theta_1\theta_5\theta_3\theta_6\theta_4V_0, \\
b_3 \quad = \quad & V_5 - (\theta_1 + \theta_2 + \theta_4 + \theta_5 + \theta_6)V_4 + (\theta_1\theta_2 + \theta_1\theta_5 + \theta_1\theta_6 + \theta_2\theta_5 + \theta_1\theta_4 + \theta_2\theta_6 + \theta_5\theta_6 + \theta_2\theta_4 + \theta_5\theta_4 + \theta_6\theta_4)V_3 \\
& -(\theta_1\theta_2\theta_5 + \theta_1\theta_2\theta_6 + \theta_1\theta_5\theta_6 + \theta_1\theta_2\theta_4 + \theta_1\theta_5\theta_4 + \theta_2\theta_5\theta_6 + \theta_1\theta_6\theta_4 + \theta_2\theta_5\theta_4 + \theta_2\theta_6\theta_4 + \theta_5\theta_6\theta_4)V_2 \\
& +(\theta_1\theta_2\theta_5\theta_6 + \theta_1\theta_2\theta_5\theta_4 + \theta_1\theta_2\theta_6\theta_4 + \theta_1\theta_5\theta_6\theta_4 + \theta_2\theta_5\theta_6\theta_4)V_1 - \theta_1\theta_2\theta_5\theta_6\theta_4V_0, \\
b_4 \quad = \quad & V_5 - (\theta_1 + \theta_2 + \theta_3 + \theta_5 + \theta_6)V_4 + (\theta_1\theta_2 + \theta_1\theta_5 + \theta_1\theta_3 + \theta_1\theta_6 + \theta_2\theta_5 + \theta_2\theta_3 + \theta_2\theta_6 + \theta_5\theta_3 + \theta_5\theta_6 + \theta_3\theta_6)V_3 \\
& -(\theta_1\theta_2\theta_5 + \theta_1\theta_2\theta_3 + \theta_1\theta_2\theta_6 + \theta_1\theta_5\theta_3 + \theta_1\theta_5\theta_6 + \theta_1\theta_3\theta_6 + \theta_2\theta_5\theta_3 + \theta_2\theta_5\theta_6 + \theta_2\theta_3\theta_6 + \theta_5\theta_3\theta_6)V_2 \\
& +(\theta_1\theta_2\theta_5\theta_3 + \theta_1\theta_2\theta_5\theta_6 + \theta_1\theta_2\theta_3\theta_6 + \theta_1\theta_5\theta_3\theta_6 + \theta_2\theta_5\theta_3\theta_6)V_1 - \theta_1\theta_2\theta_5\theta_3\theta_6V_0, \\
b_5 \quad = \quad & V_5 - (\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_6)V_4 + (\theta_1\theta_2 + \theta_1\theta_3 + \theta_1\theta_6 + \theta_1\theta_4 + \theta_2\theta_3 + \theta_2\theta_6 + \theta_2\theta_4 + \theta_3\theta_6 + \theta_3\theta_4 + \theta_6\theta_4)V_3 \\
& -(\theta_1\theta_2\theta_3 + \theta_1\theta_2\theta_6 + \theta_1\theta_2\theta_4 + \theta_1\theta_3\theta_6 + \theta_1\theta_3\theta_4 + \theta_1\theta_6\theta_4 + \theta_2\theta_3\theta_6 + \theta_2\theta_3\theta_4 + \theta_2\theta_6\theta_4 + \theta_3\theta_6\theta_4)V_2 \\
& +(\theta_1\theta_2\theta_3\theta_6 + \theta_1\theta_2\theta_3\theta_4 + \theta_1\theta_2\theta_6\theta_4 + \theta_1\theta_3\theta_6\theta_4 + \theta_2\theta_3\theta_6\theta_4)V_1 - \theta_1\theta_2\theta_3\theta_6\theta_4V_0, \\
b_6 \quad = \quad & V_5 - (\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5)V_4 + (\theta_1\theta_2 + \theta_1\theta_5 + \theta_1\theta_3 + \theta_2\theta_5 + \theta_1\theta_4 + \theta_2\theta_3 + \theta_5\theta_3 + \theta_2\theta_4 + \theta_5\theta_4 + \theta_3\theta_4)V_3 \\
& -(\theta_1\theta_2\theta_5 + \theta_1\theta_2\theta_3 + \theta_1\theta_5\theta_3 + \theta_1\theta_2\theta_4 + \theta_1\theta_5\theta_4 + \theta_2\theta_5\theta_3 + \theta_1\theta_3\theta_4 + \theta_2\theta_5\theta_4 + \theta_2\theta_3\theta_4 + \theta_5\theta_3\theta_4)V_2 \\
& +(\theta_1\theta_2\theta_5\theta_3 + \theta_1\theta_2\theta_5\theta_4 + \theta_1\theta_2\theta_3\theta_4 + \theta_1\theta_5\theta_3\theta_4 + \theta_2\theta_5\theta_3\theta_4)V_1 - \theta_1\theta_2\theta_5\theta_3\theta_4V_0.
\end{aligned}$$

Here, $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ and θ_6 are the roots of the equation

$$x^6 - 2x^5 - 3x^4 - 5x^3 - 7x^2 - 11x - 13 = 0. \tag{1.6}$$

Moreover, the approximate value of the roots $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ and θ_6 of Equation (1.6) are given by

$$\begin{aligned}
\theta_1 &= 3.515372711921757, \\
\theta_2 &= -1.183212731145181, \\
\theta_3 &= -0.7228110394202282 + 1.063369120765496i, \\
\theta_4 &= -0.7228110394202282 - 1.063369120765496i, \\
\theta_5 &= 0.5567310490319399 + 1.257207305141223i, \\
\theta_6 &= 0.5567310490319399 - 1.257207305141223i.
\end{aligned}$$

The first few generalized 6-primes numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized 6-primes numbers

n	V_n	V_{-n}
0	V_0	
1	V_1	$\frac{1}{13}V_5 - \frac{7}{13}V_1 - \frac{5}{13}V_2 - \frac{3}{13}V_3 - \frac{2}{13}V_4 - \frac{11}{13}V_0$
2	V_2	$\frac{30}{169}V_0 + \frac{12}{169}V_1 + \frac{16}{169}V_2 + \frac{7}{169}V_3 + \frac{35}{169}V_4 - \frac{11}{169}V_5$
3	V_3	$\frac{365}{2197}V_3 - \frac{2}{2197}V_1 - \frac{59}{2197}V_2 - \frac{174}{2197}V_0 - \frac{203}{2197}V_4 + \frac{30}{2197}V_5$
4	V_4	$\frac{1888}{28\,561}V_0 + \frac{451}{28\,561}V_1 + \frac{5615}{28\,561}V_2 - \frac{2117}{28\,561}V_3 + \frac{738}{28\,561}V_4 - \frac{174}{28\,561}V_5$
5	V_5	$\frac{59\,779}{371\,293}V_1 - \frac{14\,905}{371\,293}V_0 - \frac{36\,961}{371\,293}V_2 + \frac{3930}{371\,293}V_3 - \frac{6038}{371\,293}V_4 + \frac{1888}{371\,293}V_5$
6	$13V_0 + 11V_1 + 7V_2 + 5V_3 + 3V_4 + 2V_5$	$\frac{941\,082}{4826\,809}V_0 - \frac{376\,158}{4826\,809}V_1 + \frac{125\,615}{4826\,809}V_2 - \frac{33\,779}{4826\,809}V_3 + \frac{54\,354}{4826\,809}V_4 - \frac{14\,905}{4826\,809}V_5$

Now, we define three special cases of the sequence $\{V_n\}$. 6-primes sequence$\{G_n\}_{n\geq 0}$, Lucas 6-primes sequence $\{H_n\}_{n\geq 0}$ and modified 6-primes sequence $\{E_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$\begin{aligned}
G_{n+6} &= 2G_{n+5} + 3G_{n+4} + 5G_{n+3} + 7G_{n+2} + 11G_{n+1} + 13G_n, \quad G_0 = 0, G_1 = 0, G_2 = 0, G_3 = 0, G_4 = 1, G_5 = 2, \\
H_{n+6} &= 2H_{n+5} + 3H_{n+4} + 5H_{n+3} + 7H_{n+2} + 11H_{n+1} + 13H_n, \quad H_0 = 6, H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 150, H_5 = 542, \\
E_{n+6} &= 2E_{n+5} + 3E_{n+4} + 5E_{n+3} + 7E_{n+2} + 11E_{n+1} + 13E_n, \quad E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 0, E_4 = 1, E_5 = 1.
\end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{E_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n} = -\frac{11}{13}G_{-(n-1)} - \frac{7}{13}G_{-(n-2)} - \frac{5}{13}G_{-(n-3)} - \frac{3}{13}G_{-(n-4)} - \frac{2}{13}G_{-(n-5)} + \frac{1}{13}G_{-(n-6)}, \quad (1.7)$$

$$H_{-n} = -\frac{11}{13}H_{-(n-1)} - \frac{7}{13}H_{-(n-2)} - \frac{5}{13}H_{-(n-3)} - \frac{3}{13}H_{-(n-4)} - \frac{2}{13}H_{-(n-5)} + \frac{1}{13}H_{-(n-6)} \quad (1.8)$$

and

$$E_{-n} = -\frac{11}{13}E_{-(n-1)} - \frac{7}{13}E_{-(n-2)} - \frac{5}{13}E_{-(n-3)} - \frac{3}{13}E_{-(n-4)} - \frac{2}{13}E_{-(n-5)} + \frac{1}{13}E_{-(n-6)} \quad (1.9)$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.7), (1.8) and (1.9) hold for all integers n.

Note that the sequences G_n, H_n and E_n are not indexed in [10] yet. Next, we present the first few values of the 6-primes, Lucas 6-primes and modified 6-primes numbers with positive and negative subscripts:

Table 2. The first few values of the special sisth-order numbers with positive and negative subscripts

n	0	1	2	3	4	5	6	7	8	9
G_n	0	0	0	0	1	2	7	25	88	311
G_{-n}	....	0	$\frac{1}{13}$	$-\frac{11}{169}$	$\frac{30}{2197}$	$-\frac{174}{28561}$	$\frac{1888}{371293}$	$-\frac{14905}{4826809}$	$\frac{941082}{62748517}$	$-\frac{15241956}{815730721}$
H_n	6	2	10	41	150	542	1909	6617	23302	81977
H_{-n}	....	$-\frac{11}{13}$	$-\frac{61}{169}$	$-\frac{863}{2197}$	$-\frac{2025}{28561}$	$-\frac{60756}{371293}$	$\frac{4839977}{4826809}$	$-\frac{40\,911574}{62748517}$	$\frac{100922415}{815730721}$	$-\frac{1281284909}{10\,604499373}$
E_n	0	0	0	0	1	1	5	18	63	223
E_{-n}	....	$-\frac{1}{13}$	$\frac{24}{169}$	$-\frac{173}{2197}$	$\frac{564}{28561}$	$-\frac{4150}{371293}$	$\frac{39449}{4826809}$	$-\frac{1134847}{62748517}$	$\frac{27476\,022}{815730721}$	$-\frac{301397417}{10\,604499373}$

For all integers n, 6-primes, Lucas 6-primes and modified 6-primes numbers (using initial conditions in (1.5)) can be expressed using Binet's formulas as

$$\begin{aligned}
G_n &= \sum_{k=1}^{6} \frac{\theta_k^{n+1}}{\prod\limits_{\substack{j=1 \\ k\neq j}}^{6} (\theta_k - \theta_j)}, \\
H_n &= \sum_{k=1}^{6} \theta_k^n = \theta_1^n + \theta_2^n + \theta_3^n + \theta_4^n + \theta_5^n + \theta_6^n, \\
E_n &= \sum_{k=1}^{6} \frac{(\theta_k - 1)\theta_k^{n+1}}{\prod\limits_{\substack{j=1 \\ k\neq j}}^{6} (\theta_k - \theta_j)},
\end{aligned}$$

respectively.

2 GENERATING FUNCTIONS AND OBTAINING BINET FORMULA FROM GENERATING FUNCTION

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} V_n x^n$ of the sequence V_n. The following lemma is a special case of a well known formula of generating functions of the generalized m-step Fibonacci numbers which can be found in the literature (see for example [11]).

Lemma 2.1. *Suppose that $f_{V_n}(x) = \sum\limits_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized 6-primes sequence $\{V_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} V_n x^n$ is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{\Lambda}{1 - 2x - 3x^2 - 5x^3 - 7x^4 - 11x^5 - 13x^6}. \tag{2.1}$$

where

$$\begin{aligned}
\Lambda &= V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - 3V_0)x^2 + (V_3 - 2V_2 - 3V_1 - 5V_0)x^3 \\
&\quad + (V_4 - 2V_3 - 3V_2 - 5V_1 - 7V_0)x^4 + (V_5 - 2V_4 - 3V_3 - 5V_2 - 7V_1 - 11V_0)x^5 \\
&= V_0 + \sum_{i=1}^{6-1} x^i \left(V_i - \sum_{j=1}^{i} r_j V_{i-j} \right).
\end{aligned}$$

The previous lemma gives the following results as particular examples.

Corollary 2.1. Generated functions of 6-primes, Lucas 6-primes and modified 6-primes numbers are

$$\sum_{n=0}^{\infty} H_n x^n = \frac{x^4}{1-2x-3x^2-5x^3-7x^4-11x^5-13x^6},$$

and

$$\sum_{n=0}^{\infty} H_n x^n = \frac{6-10x-12x^2-15x^3-14x^4-11x^5}{1-2x-3x^2-5x^3-7x^4-11x^5-13x^6},$$

and

$$\sum_{n=0}^{\infty} E_n x^n = \frac{x^4-x^5}{1-2x-3x^2-5x^3-7x^4-11x^5-13x^6},$$

respectively.

We next find Binet's formula of generalized 6-primes numbers $\{V_n\}$ by the use of generating function for V_n.

Theorem 2.1. *(Binet's formula of generalized 6-primes numbers)*

$$V_n = \sum_{k=1}^{6} \frac{d_k \theta_k^n}{\prod\limits_{\substack{j=1 \\ k\neq j}}^{6} (\theta_k - \theta_j)} \tag{2.2}$$

where

$$\begin{aligned} d_1 &= V_0\theta_1^{6-1} + \sum_{i=1}^{6-1} \theta_1^{6-1-i}\left[V_i - \sum_{j=1}^{i} r_j V_{i-j}\right], \\ d_l &= V_0\theta_l^{6-1} + \sum_{i=1}^{6-1} \theta_l^{6-1-i}\left[V_i - \sum_{j=1}^{i} r_j V_{i-j}\right], \quad 1 \le l \le m = 6, \\ r_1 &= 2, r_2 = 3, r_3 = 5, r_4 = 7, r_5 = 11, r_6 = 13. \end{aligned}$$

Proof. Let

$$h(x) = 1-2x-3x^2-5x^3-7x^4-11x^5-13x^6.$$

Then for some $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ and θ_6 we write

$$h(x) = (1-\theta_1 x)(1-\theta_2 x)(1-\theta_3 x)(1-\theta_4 x)(1-\theta_5 x)(1-\theta_6 x)$$

i.e.,

$$1-2x-3x^2-5x^3-7x^4-11x^5-13x^6 = (1-\theta_1 x)(1-\theta_2 x)(1-\theta_3 x)(1-\theta_4 x)(1-\theta_5 x)(1-\theta_6 x). \tag{2.3}$$

Hence $\frac{1}{\theta_1}, \frac{1}{\theta_2}, \frac{1}{\theta_3}, \frac{1}{\theta_4}, \frac{1}{\theta_5}$ and $\frac{1}{\theta_6}$ are the roots of $h(x)$. This gives $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ and θ_6 as the roots of

$$h(\frac{1}{x}) = 1 - \frac{2}{x} - \frac{3}{x^2} - \frac{5}{x^3} - \frac{7}{x^4} - \frac{11}{x^5} - \frac{13}{x^6} = 0.$$

This implies $x^6 - 2x^5 - 3x^4 - 5x^3 - 7x^2 - 11x - 13 = 0$. Now, by (2.1) and (2.3), it follows that

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + \sum_{i=1}^{6-1} x^i \left[V_i - \sum_{j=1}^{i} r_j V_{i-j}\right]}{(1-\theta_1 x)(1-\theta_2 x)(1-\theta_3 x)(1-\theta_4 x)(1-\theta_5 x)(1-\theta_6 x)}.$$

Then we write

$$\frac{V_0+\sum_{i=1}^{6-1}x^i\left[V_i-\sum_{j=1}^{i}r_jV_{i-j}\right]}{(1-\theta_1x)(1-\theta_2x)(1-\theta_3x)(1-\theta_4x)(1-\theta_5x)(1-\theta_6x)}=\frac{A_1}{(1-\theta_1x)}+\frac{A_2}{(1-\theta_2x)}+\frac{A_3}{(1-\theta_3x)}+\frac{A_4}{(1-\theta_4x)}+\frac{A_5}{(1-\theta_5x)}+\frac{A_6}{(1-\theta_6x)}. \tag{2.4}$$

So

$$\begin{aligned} & V_0+\sum_{i=1}^{6-1}x^i\left[V_i-\sum_{j=1}^{i}r_jV_{i-j}\right] \\ = \; & A_1(1-\theta_2x)(1-\theta_3x)(1-\theta_4x)(1-\theta_5x)(1-\theta_6x)+A_2(1-\theta_1x)(1-\theta_3x)(1-\theta_4x)(1-\theta_5x)(1-\theta_6x) \\ & +A_3(1-\theta_1x)(1-\theta_2x)(1-\theta_4x)(1-\theta_5x)(1-\theta_6x)+A_4(1-\theta_1x)(1-\theta_2x)(1-\theta_3x)(1-\theta_5x)(1-\theta_6x) \\ & +A_5(1-\theta_1x)(1-\theta_2x)(1-\theta_3x)(1-\theta_4x)(1-\theta_6x)+A_6(1-\theta_1x)(1-\theta_2x)(1-\theta_3x)(1-\theta_4x)(1-\theta_5x). \end{aligned}$$

If we consider $x=\frac{1}{\theta_1}$, we get

$$V_0+\sum_{i=1}^{6-1}\left(\frac{1}{\theta_1}\right)^i\left[V_i-\sum_{j=1}^{i}r_jV_{i-j}\right]=A_1(1-\frac{\theta_2}{\theta_1})(1-\frac{\theta_3}{\theta_1})(1-\frac{\theta_4}{\theta_1})(1-\frac{\theta_5}{\theta_1})(1-\frac{\theta_6}{\theta_1}).$$

This gives

$$\begin{aligned} A_1 &= \frac{\theta_1^5(V_0+\sum_{i=1}^{6-1}\left(\frac{1}{\theta_1}\right)^i\left[V_i-\sum_{j=1}^{i}r_jV_{i-j}\right])}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-\theta_4)(\theta_1-\theta_5)(\theta_1-\theta_6)} \\ &= \frac{V_0\theta_1^{6-1}+\sum_{i=1}^{6-1}\theta_1^{6-1-i}\left[V_i-\sum_{j=1}^{i}r_jV_{i-j}\right]}{(\theta_1-\theta_2)(\theta_1-\theta_3)(\theta_1-\theta_4)(\theta_1-\theta_5)(\theta_1-\theta_6)}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} A_2 &= \frac{V_0\theta_2^{6-1}+\sum_{i=1}^{6-1}\theta_2^{6-1-i}\left[V_i-\sum_{j=1}^{i}r_jV_{i-j}\right]}{(\theta_2-\theta_1)(\theta_2-\theta_3)(\theta_2-\theta_4)(\theta_2-\theta_5)(\theta_2-\theta_6)}, \\ A_3 &= \frac{V_0\theta_3^{6-1}+\sum_{i=1}^{6-1}\theta_3^{6-1-i}\left[V_i-\sum_{j=1}^{i}r_jV_{i-j}\right]}{(\theta_3-\theta_1)(\theta_3-\theta_2)(\theta_3-\theta_4)(\theta_3-\theta_5)(\theta_3-\theta_6)}, \\ A_4 &= \frac{V_0\theta_4^{6-1}+\sum_{i=1}^{6-1}\theta_4^{6-1-i}\left[V_i-\sum_{j=1}^{i}r_jV_{i-j}\right]}{(\theta_4-\theta_1)(\theta_4-\theta_2)(\theta_4-\theta_3)(\theta_4-\theta_5)(\theta_4-\theta_6)}, \\ A_5 &= \frac{V_0\theta_5^{6-1}+\sum_{i=1}^{6-1}\theta_5^{6-1-i}\left[V_i-\sum_{j=1}^{i}r_jV_{i-j}\right]}{(\theta_5-\theta_1)(\theta_5-\theta_2)(\theta_5-\theta_3)(\theta_5-\theta_4)(\theta_5-\theta_6)}, \\ A_6 &= \frac{V_0\theta_6^{6-1}+\sum_{i=1}^{6-1}\theta_6^{6-1-i}\left[V_i-\sum_{j=1}^{i}r_jV_{i-j}\right]}{(\theta_6-\theta_1)(\theta_6-\theta_2)(\theta_6-\theta_3)(\theta_6-\theta_4)(\theta_6-\theta_5)}. \end{aligned}$$

Thus (2.4) can be written as

$$\sum_{n=0}^{\infty}V_nx^n=A_1(1-\theta_1x)^{-1}+A_2(1-\theta_2x)^{-1}+A_3(1-\theta_3x)^{-1}+A_4(1-\theta_4x)^{-1}+A_5(1-\theta_5x)^{-1}+A_6(1-\theta_6x)^{-1}.$$

This gives

$$\begin{aligned} \sum_{n=0}^{\infty}V_nx^n &= A_1\sum_{n=0}^{\infty}\theta_1^nx^n+A_2\sum_{n=0}^{\infty}\theta_2^nx^n+A_3\sum_{n=0}^{\infty}\theta_3^nx^n+A_4\sum_{n=0}^{\infty}\theta_4^nx^n+A_5\sum_{n=0}^{\infty}\theta_5^nx^n+A_6\sum_{n=0}^{\infty}\theta_6^nx^n \\ &= \sum_{n=0}^{\infty}(A_1\theta_1^n+A_2\theta_2^n+A_3\theta_3^n+A_4\theta_4^n+A_5\theta_5^n+A_6\theta_6^n)x^n. \end{aligned}$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$V_n = A_1\theta_1^n + A_2\theta_2^n + A_3\theta_3^n + A_4\theta_4^n + A_5\theta_5^n + A_6\theta_6^n$$

and then we get (2.2). □

Next, using Theorem 2.1, we present the Binet's formulas of 6-primes, Lucas 6-primes and modified 6-primes sequences.

Corollary 2.2. Binet's formulas of 6-primes, Lucas 6-primes and modified 6-primes sequences are

$$\begin{aligned} G_n &= \sum_{k=1}^{6} \frac{\theta_k^{n+1}}{\prod\limits_{\substack{j=1 \\ k\neq j}}^{6} (\theta_k - \theta_j)}, \\ H_n &= \sum_{k=1}^{6} \theta_k^n = \theta_1^n + \theta_2^n + \theta_3^n + \theta_4^n + \theta_5^n + \theta_6^n, \\ E_n &= \sum_{k=1}^{6} \frac{(\theta_k - 1)\theta_k^{n+1}}{\prod\limits_{\substack{j=1 \\ k\neq j}}^{6} (\theta_k - \theta_j)}, \end{aligned}$$

respectively.

3 SIMSON FORMULAS

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized 6-primes sequence $\{V_n\}_{n\geq 0}$.

Theorem 3.1 (Simson Formula of Generalized 6-primes Numbers)**.** *For all integers n, we have*

$$\begin{vmatrix} V_{n+5} & V_{n+4} & V_{n+3} & V_{n+2} & V_{n+1} & V_n \\ V_{n+4} & V_{n+3} & V_{n+2} & V_{n+1} & V_n & V_{n-1} \\ V_{n+3} & V_{n+2} & V_{n+1} & V_n & V_{n-1} & V_{n-2} \\ V_{n+2} & V_{n+1} & V_n & V_{n-1} & V_{n-2} & V_{n-3} \\ V_{n+1} & V_n & V_{n-1} & V_{n-2} & V_{n-3} & V_{n-4} \\ V_n & V_{n-1} & V_{n-2} & V_{n-3} & V_{n-4} & V_{n-5} \end{vmatrix} = (-1)^n 13^n \begin{vmatrix} V_5 & V_4 & V_3 & V_2 & V_1 & V_0 \\ V_4 & V_3 & V_2 & V_1 & V_0 & V_{-1} \\ V_3 & V_2 & V_1 & V_0 & V_{-1} & V_{-2} \\ V_2 & V_1 & V_0 & V_{-1} & V_{-2} & V_{-3} \\ V_1 & V_0 & V_{-1} & V_{-2} & V_{-3} & V_{-4} \\ V_0 & V_{-1} & V_{-2} & V_{-3} & V_{-4} & V_{-5} \end{vmatrix}.$$

Proof. It is given in Soykan [12]. □

The previous theorem gives the following results as particular examples.

Corollary 3.1. For all integers n, Simson formula of 6-primes, Lucas 6-primes and modified 6-primes numbers are given as

$$\begin{vmatrix} G_{n+5} & G_{n+4} & G_{n+3} & G_{n+2} & G_{n+1} & G_n \\ G_{n+4} & G_{n+3} & G_{n+2} & G_{n+1} & G_n & G_{n-1} \\ G_{n+3} & G_{n+2} & G_{n+1} & G_n & G_{n-1} & G_{n-2} \\ G_{n+2} & G_{n+1} & G_n & G_{n-1} & G_{n-2} & G_{n-3} \\ G_{n+1} & G_n & G_{n-1} & G_{n-2} & G_{n-3} & G_{n-4} \\ G_n & G_{n-1} & G_{n-2} & G_{n-3} & G_{n-4} & G_{n-5} \end{vmatrix} = (-1)^{n+1}13^{n-4} \quad (3.1)$$

and

$$\begin{vmatrix} H_{n+5} & H_{n+4} & H_{n+3} & H_{n+2} & H_{n+1} & H_n \\ H_{n+4} & H_{n+3} & H_{n+2} & H_{n+1} & H_n & H_{n-1} \\ H_{n+3} & H_{n+2} & H_{n+1} & H_n & H_{n-1} & H_{n-2} \\ H_{n+2} & H_{n+1} & H_n & H_{n-1} & H_{n-2} & H_{n-3} \\ H_{n+1} & H_n & H_{n-1} & H_{n-2} & H_{n-3} & H_{n-4} \\ H_n & H_{n-1} & H_{n-2} & H_{n-3} & H_{n-4} & H_{n-5} \end{vmatrix} = 99191747 \times 2^5 \times 41 \times (-1)^{n+1}13^{n-5} \quad (3.2)$$

and

$$\begin{vmatrix} E_{n+5} & E_{n+4} & E_{n+3} & E_{n+2} & E_{n+1} & E_n \\ E_{n+4} & E_{n+3} & E_{n+2} & E_{n+1} & E_n & E_{n-1} \\ E_{n+3} & E_{n+2} & E_{n+1} & E_n & E_{n-1} & E_{n-2} \\ E_{n+2} & E_{n+1} & E_n & E_{n-1} & E_{n-2} & E_{n-3} \\ E_{n+1} & E_n & E_{n-1} & E_{n-2} & E_{n-3} & E_{n-4} \\ E_n & E_{n-1} & E_{n-2} & E_{n-3} & E_{n-4} & E_{n-5} \end{vmatrix} = 5 \times 2^3 \times (-1)^{n+1}13^{n-5} \quad (3.3)$$

respectively.

4 SOME IDENTITIES

In this section, we obtain some identities of 6-primes, Lucas 6-primes and modified 6-primes numbers. First, we can give a few basic relations between $\{G_n\}$ and $\{H_n\}$.

Lemma 4.1. *The following equalities are true:*

$$\begin{aligned} 169H_n &= -61G_{n+6} - 21G_{n+5} + 1483G_{n+4} - 956G_{n+3} - 886G_{n+2} - 863G_{n+1}, \quad (4.1) \\ 13H_n &= -11G_{n+5} + 100G_{n+4} - 97G_{n+3} - 101G_{n+2} - 118G_{n+1} - 61G_n, \\ H_n &= 6G_{n+4} - 10G_{n+3} - 12G_{n+2} - 15G_{n+1} - 14G_n - 11G_{n-1}, \\ H_n &= 2G_{n+3} + 6G_{n+2} + 15G_{n+1} + 28G_n + 55G_{n-1} + 78G_{n-2}, \\ H_n &= 10G_{n+2} + 21G_{n+1} + 38G_n + 69G_{n-1} + 100G_{n-2} + 26G_{n-3}, \\ H_n &= 41G_{n+1} + 68G_n + 119G_{n-1} + 170G_{n-2} + 136G_{n-3} + 130G_{n-4}, \end{aligned}$$

and

$$\begin{aligned}
65069786032G_n &= 17165493H_{n+6} - 48224301H_{n+5} + 84682036H_{n+4} - 663056677H_{n+3} \\
&\quad +802372816H_{n+2} - 86032399H_{n+1}, \\
65069786032G_n &= -13893315H_{n+5} + 136178515H_{n+4} - 577229212H_{n+3} + 922531267H_{n+2} \\
&\quad +102788024H_{n+1} + 223151409H_n, \\
65069786032G_n &= 108391885H_{n+4} - 618909157H_{n+3} + 853064692H_{n+2} + 5534819H_{n+1} \\
&\quad +70324944H_n - 180613095H_{n-1}, \\
65069786032G_n &= -402125387H_{n+3} + 1178240347H_{n+2} + 547494244H_{n+1} + 829068139H_n \\
&\quad +1011697640H_{n-1} + 1409094505H_{n-2}, \\
65069786032G_n &= 373989573H_{n+2} - 658881917H_{n+1} - 1181558796H_n - 1803180069H_{n-1} \\
&\quad -3014284752H_{n-2} - 5227630031H_{n-3}, \\
65069786032G_n &= 89097229H_{n+1} - 59590077H_n + 66767796H_{n-1} - 396\,357741H_{n-2} \\
&\quad -1113744728H_{n-3} + 4861864449H_{n-4}.
\end{aligned}$$

Proof. Note that all the identities hold for all integers n. We prove (4.1). To show (4.1), writing

$$H_n = a \times G_{n+6} + b \times G_{n+5} + c \times G_{n+4} + d \times G_{n+3} + e \times G_{n+2} + f \times G_{n+1}$$

and solving the system of equations

$$\begin{aligned}
H_0 &= a \times G_6 + b \times G_5 + c \times G_4 + d \times G_3 + e \times G_2 + f \times G_1 \\
H_1 &= a \times G_7 + b \times G_6 + c \times G_5 + d \times G_4 + e \times G_3 + f \times G_2 \\
H_2 &= a \times G_8 + b \times G_7 + c \times G_6 + d \times G_5 + e \times G_4 + f \times G_3 \\
H_3 &= a \times G_9 + b \times G_8 + c \times G_7 + d \times G_6 + e \times G_5 + f \times G_4 \\
H_4 &= a \times G_{10} + b \times G_9 + c \times G_8 + d \times G_7 + e \times G_6 + f \times G_5 \\
H_5 &= a \times G_{11} + b \times G_{10} + c \times G_9 + d \times G_8 + e \times G_7 + f \times G_6
\end{aligned}$$

we find that $a = -\frac{61}{169}, b = -\frac{21}{169}, c = \frac{1483}{169}, d = -\frac{956}{169}, e = -\frac{886}{169}, f = -\frac{863}{169}$. The other equalities can be proved similarly. $\square$

Secondly, we present a few basic relations between $\{G_n\}$ and $\{E_n\}$.

Lemma 4.2. *The following equalities are true:*

$$\begin{aligned}
169E_n &= 24G_{n+6} - 61G_{n+5} - 46G_{n+4} - 81G_{n+3} - 103G_{n+2} - 173G_{n+1}, \\
13E_n &= -G_{n+5} + 2G_{n+4} + 3G_{n+3} + 5G_{n+2} + 7G_{n+1} + 24G_n, \\
E_n &= G_n - G_{n-1},
\end{aligned}$$

and

$$\begin{aligned}
40G_n &= E_{n+6} - E_{n+5} - 4E_{n+4} - 9E_{n+3} - 16E_{n+2} - 27E_{n+1}, \\
40G_n &= E_{n+5} - E_{n+4} - 4E_{n+3} - 9E_{n+2} - 16E_{n+1} + 13E_n, \\
40G_n &= E_{n+4} - E_{n+3} - 4E_{n+2} - 9E_{n+1} + 24E_n + 13E_{n-1}, \\
40G_n &= E_{n+3} - E_{n+2} - 4E_{n+1} + 31E_n + 24E_{n-1} + 13E_{n-2}, \\
40G_n &= E_{n+2} - E_{n+1} + 36E_n + 31E_{n-1} + 24E_{n-2} + 13E_{n-3}, \\
40G_n &= E_{n+1} + 39E_n + 36E_{n-1} + 31E_{n-2} + 24E_{n-3} + 13E_{n-4}.
\end{aligned}$$

Note that all the identities in the above lemma can be proved by induction as well.

Thirdly, we give a few basic relations between $\{H_n\}$ and $\{E_n\}$.

Lemma 4.3. *The following equalities are true:*

$$\begin{aligned}
65H_n &= -36E_{n+6} - 19E_{n+5} + 589E_{n+4} + 284E_{n+3} + 31E_{n+2} - 163E_{n+1},\\
5H_n &= -7E_{n+5} + 37E_{n+4} + 8E_{n+3} - 17E_{n+2} - 43E_{n+1} - 36E_n,\\
5H_n &= 23E_{n+4} - 13E_{n+3} - 52E_{n+2} - 92E_{n+1} - 113E_n - 91E_{n-1},\\
5H_n &= 33E_{n+3} + 17E_{n+2} + 23E_{n+1} + 48E_n + 162E_{n-1} + 299E_{n-2},\\
5H_n &= 83E_{n+2} + 122E_{n+1} + 213E_n + 393E_{n-1} + 662E_{n-2} + 429E_{n-3},\\
5H_n &= 288E_{n+1} + 462E_n + 808E_{n-1} + 1243E_{n-2} + 1342E_{n-3} + 1079E_{n-4},
\end{aligned}$$

and

$$\begin{aligned}
105738402302E_n &= 38647976H_{n+6} - 127766515H_{n+5} + 183710648H_{n+4} - 1268845658H_{n+3}\\
&\quad +2306044577H_{n+2} - 1561953023H_{n+1},\\
8133723254E_n &= -3882351H_{n+5} + 23050352H_{n+4} - 82738906H_{n+3} + 198198493H_{n+2}\\
&\quad -87448099H_{n+1} + 38647976H_n,\\
8133723254E_n &= 15285650H_{n+4} - 94385959H_{n+3} + 178786738H_{n+2} - 114624556H_{n+1}\\
&\quad -4057885H_n - 50470563H_{n-1},\\
8133723254E_n &= -63814659H_{n+3} + 224643688H_{n+2} - 38196306H_{n+1} + 102941665H_n\\
&\quad +117671587H_{n-1} + 198713450H_{n-2},\\
8133723254E_n &= 97014370H_{n+2} - 229640283H_{n+1} - 216131630H_n - 329031026H_{n-1}\\
&\quad -503247799H_{n-2} - 829590567H_{n-3},\\
8133723254E_n &= -35611543H_{n+1} + 74911480H_n + 156040824H_{n-1} + 175852791H_{n-2}\\
&\quad +237567503H_{n-3} + 1261186810H_{n-4}.
\end{aligned}$$

We now present a few special identities for the modified 6-primes sequence $\{E_n\}$.

Theorem 4.1. *(Catalan's identity) For all integers n and m, the following identity holds*

$$\begin{aligned}
E_{n+m}E_{n-m} - E_n^2 &= (G_{n+m} - G_{n+m-1})(G_{n-m} - G_{n-m-1}) - (G_n - G_{n-1})^2\\
&= (G_n(G_m - G_{m+1}) + G_{n-1}(-G_m + G_{m-2}) + G_{n-2}(-G_m + G_{m-1}))\\
&\quad (G_n(G_{-m} - G_{1-m}) + G_{n-1}(-G_{-m} + G_{-m-2}) + G_{n-2}(-G_{-m} + G_{-m-1}))\\
&\quad -(G_n - G_{n-1})^2.
\end{aligned}$$

Proof. We use the identity

$$E_n = G_n - G_{n-1}.$$

□

Note that for $m = 1$ in Catalan's identity, we get the Cassini's identity for the modified 6-primes sequnce.

Corollary 4.1. (Cassini's identity) For all integers numbers n and m, the following identity holds

$$E_{n+1}E_{n-1} - E_n^2 = (G_{n+1} - G_n)(G_{n-1} - G_{n-2}) - (G_n - G_{n-1})^2.$$

The d'Ocagne's, Gelin-Cesàro's and Melham's identities can also be obtained by using $E_n = G_n - G_{n-1}$. The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham's identities of modified 6-primes sequence $\{E_n\}$.

Theorem 4.2. *Let n and m be any integers. Then the following identities are true:*

(a) *(d'Ocagne's identity)*

$$E_{m+1}E_n - E_mE_{n+1} = (G_{m+1} - G_m)(G_n - G_{n-1}) - (G_m - G_{m-1})(G_{n+1} - G_n).$$

(b) *(Gelin-Cesàro's identity)*

$$E_{n+2}E_{n+1}E_{n-1}E_{n-2} - E_n^4 = (G_{n+2} - G_{n+1})(G_{n+1} - G_n)(G_{n-1} - G_{n-2})(G_{n-2} - G_{n-3}) - (G_n - G_{n-1})^4.$$

(c) *(Melham's identity)*

$$E_{n+1}E_{n+2}E_{n+6} - E_{n+3}^3 = (G_{n+1} - G_n)(G_{n+2} - G_{n+1})(G_{n+6} - G_{n+5}) - (G_{n+3} - G_{n+2})^3.$$

Proof. Use the identity $E_n = G_n - G_{n-1}$. □

5 LINEAR SUMS

The following theorem presents some linear summing formulas of generalized Hexanacci numbers with positive subscripts.

Theorem 5.1. *For $n \geq 0$, we have the following formulas:*

(a) *(Sum of the generalized Hexanacci numbers) If $r_1 + r_2 + r_3 + r_4 + r_5 + r_6 - 1 \neq 0$ then*

$$\sum_{k=0}^{n} W_k = \frac{\Delta_1}{r_1 + r_2 + r_3 + r_4 + r_5 + r_6 - 1}$$

where

$$\begin{aligned}
\Delta_1 \quad = \quad & W_{n+6} + (1 - r_1)W_{n+5} + (1 - r_1 - r_2)W_{n+4} + (1 - r_1 - r_2 - r_3)W_{n+3} \\
& + (1 - r_1 - r_2 - r_3 - r_4)W_{n+2} + (1 - r_1 - r_2 - r_3 - r_4 - r_5)W_{n+1} \\
& - W_5 + (r_1 - 1)W_4 + (r_1 + r_2 - 1)W_3 + (r_1 + r_2 + r_3 - 1)W_2 \\
& + (r_1 + r_2 + r_3 + r_4 - 1)W_1 + (r_1 + r_2 + r_3 + r_4 + r_5 - 1)W_0.
\end{aligned}$$

(b) *If $(r_1 + r_2 + r_3 + r_4 + r_5 + r_6 - 1)(r_1 - r_2 + r_3 - r_4 + r_5 - r_6 + 1) \neq 0$ then*

$$\sum_{k=0}^{n} W_{2k} = \frac{\Delta_2}{(r_1 + r_2 + r_3 + r_4 + r_5 + r_6 - 1)(r_1 - r_2 + r_3 - r_4 + r_5 - r_6 + 1)}$$

where

$$\begin{aligned}
\Delta_2 \quad = \quad & -(r_2 + r_4 + r_6 - 1)W_{2n+2} + (r_3 + r_5 + r_1(r_2 + r_4 + r_6))W_{2n+1} \\
& + (r_4 + r_6 + r_1(r_3 + r_5) - r_2(r_4 + r_6) + (r_3 + r_5)^2 - (r_4 + r_6)^2)W_{2n} \\
& + (r_5 - r_2r_5 + (r_1 + r_3)r_4 + (r_1 + r_3)r_6)W_{2n-1} + (r_6 + (r_1 + r_3)r_5 - (r_2 + r_4)r_6 + r_5^2 - r_6^2)W_{2n-2} \\
& + r_6(r_1 + r_3 + r_5)W_{2n-3} - (r_1 + r_3 + r_5)W_5 + (r_2 + r_4 + r_6 + (r_1 + r_3 + r_5)r_1 - 1)W_4 \\
& + ((r_3 + r_5)r_2 - (r_4 + r_6)r_1 - r_3 - r_5)W_3 \\
& + (r_4 + r_6 + (r_1 + r_3)r_5 - (r_4 + r_6)r_2 + (r_1 + r_3)^2 - (r_2 - 1)^2)W_2 + (-r_5 + (r_2 + r_4)r_5 - (r_1 + r_3)r_6)W_1 \\
& + (2r_2 + 2r_4 + 2r_1r_5 + 2r_3r_5 - r_2r_6 + r_6 - r_4r_6 + (r_1 + r_3)^2 - (r_2 + r_4)^2 + r_5^2 - 1)W_0.
\end{aligned}$$

(c)

$$\sum_{k=0}^{n} W_{2k+1} = \frac{\Delta_3}{(r_1+r_2+r_3+r_4+r_5+r_6-1)(r_1-r_2+r_3-r_4+r_5-r_6+1)}$$

where

$$\begin{aligned}
\Delta_3 = \; & (r_1+r_3+r_5)W_{2n+2} + ((r_2+r_4+r_6) - (r_2+r_4+r_6)^2 + (r_3+r_5)^2 + r_1(r_3+r_5))W_{2n+1} \\
& + ((1-r_2)(r_3+r_5) + r_1(r_4+r_6))W_{2n} + ((r_1+r_3)r_5 + r_5^2 - (r_4+r_6)r_2 + (r_4+r_6) - (r_4+r_6)^2)W_{2n-1} \\
& + ((1-(r_2+r_4))r_5 + (r_1+r_3)r_6)W_{2n-2} - r_6(r_2+r_4+r_6-1)W_{2n-3} + (r_2+r_4+r_6-1)W_5 \\
& - ((r_3+r_5) + (r_2+r_4+r_6)r_1)W_4 + (2r_2+r_4+r_6+r_1r_3+r_1r_5-r_2r_4-r_2r_6+r_1^2-r_2^2-1)W_3 \\
& - ((1-r_2)r_5 + (r_1+r_3)(r_4+r_6))W_2 \\
& + (2r_2+2r_4+r_6+r_1r_5+r_3r_5-r_2r_6-r_4r_6+r_1^2+r_3^2+2r_1r_3-r_2^2-r_4^2-2r_2r_4-1)W_1 - r_6(r_1+r_3+r_5)W_0.
\end{aligned}$$

Proof. The proof is given in Soykan [13].

The following proposition presents some formulas of generalized 6-primes numbers with positive subscripts.

Proposition 5.1. *If* $r_1 = 2, r_2 = 3, r_3 = 5, r_4 = 7, r_5 = 11, r_6 = 13$ *then for* $n \geq 0,$ *we have the following formulas:*

(a) $\sum_{k=0}^{n} V_k = \frac{1}{40}(V_{n+6} - V_{n+5} - 4V_{n+4} - 9V_{n+3} - 16V_{n+2} - 27V_{n+1} - V_5 + V_4 + 4V_3 + 9V_2 + 16V_1 + 27V_0).$

(b) $\sum_{k=0}^{n} V_{2k} = \frac{1}{80}(11V_{2n+2} - 31V_{2n+1} + 76V_{2n} - 59V_{2n-1} + 44V_{2n-2} - 117V_{2n-3} + 9V_5 - 29V_4 + 4V_3 - 41V_2 - 4V_1 - 63V_0).$

(c) $\sum_{k=0}^{n} V_{2k+1} = \frac{1}{80}(-9V_{2n+2} + 109V_{2n+1} - 4V_{2n} + 121V_{2n-1} + 4V_{2n-2} + 143V_{2n-3} - 11V_5 + 31V_4 + 4V_3 + 59V_2 + 36V_1 + 117V_0).$

Proof. Take $r_1 = 2, r_2 = 3, r_3 = 5, r_4 = 7, r_5 = 11$ in Theorem 5.1.

As special cases of above proposition, we have the following three corollaries. First one presents some summing formulas of 6-primes numbers (take $V_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 0, G_3 = 0, G_4 = 1, G_5 = 2$).

Corollary 5.1. For $n \geq 0,$ we have the following formulas:

(a) $\sum_{k=0}^{n} G_k = \frac{1}{40}(G_{n+6} - G_{n+5} - 4G_{n+4} - 9G_{n+3} - 16G_{n+2} - 27G_{n+1} - 1).$

(b) $\sum_{k=0}^{n} G_{2k} = \frac{1}{80}(11G_{2n+2} - 31G_{2n+1} + 76G_{2n} - 59G_{2n-1} + 44G_{2n-2} - 117G_{2n-3} - 11).$

(c) $\sum_{k=0}^{n} G_{2k+1} = \frac{1}{80}(-9G_{2n+2} + 109G_{2n+1} - 4G_{2n} + 121G_{2n-1} + 4G_{2n-2} + 143G_{2n-3} + 9).$

Second one presents some summing formulas of Lucas 6-primes numbers (take $G_n = H_n$ with $H_0 = 6, H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 150, H_5 = 542$).

Corollary 5.2. For $n \geq 0,$ we have the following formulas:

(a) $\sum_{k=0}^{n} H_k = \frac{1}{40}(H_{n+6} - H_{n+5} - 4H_{n+4} - 9H_{n+3} - 16H_{n+2} - 27H_{n+1} + 56).$

(b) $\sum_{k=0}^{n} H_{2k} = \frac{1}{80}(11H_{2n+2} - 31H_{2n+1} + 76H_{2n} - 59H_{2n-1} + 44H_{2n-2} - 117H_{2n-3} - 104).$

(c) $\sum_{k=0}^{n} H_{2k+1} = \frac{1}{80}(-9H_{2n+2} + 109H_{2n+1} - 4H_{2n} + 121H_{2n-1} + 4H_{2n-2} + 143H_{2n-3} + 216).$

Third one presents some summing formulas of modified 6-primes numbers (take $H_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 0, E_4 = 1, E_5 = 1$).

Corollary 5.3. For $n \geq 0,$ we have the following formulas:

(a) $\sum_{k=0}^{n} E_k = \frac{1}{40}(E_{n+6} - E_{n+5} - 4E_{n+4} - 9E_{n+3} - 16E_{n+2} - 27E_{n+1})$.

(b) $\sum_{k=0}^{n} E_{2k} = \frac{1}{80}(11E_{2n+2} - 31E_{2n+1} + 76E_{2n} - 59E_{2n-1} + 44E_{2n-2} - 117E_{2n-3} - 20)$.

(c) $\sum_{k=0}^{n} E_{2k+1} = \frac{1}{80}(-9E_{2n+2} + 109E_{2n+1} - 4E_{2n} + 121E_{2n-1} + 4E_{2n-2} + 143E_{2n-3} + 20)$.

The following Theorem presents some linear summing formulas of generalized Hexanacci numbers with negative subscripts.

Theorem 5.2. *For $n \geq 1$, we have the following formulas:*

(a) *(Sum of the generalized Hexanacci numbers with negative indices) If $r_1+r_2+r_3+r_4+r_5+r_6-1 \neq 0$, then*

$$\sum_{k=1}^{n} W_{-k} = \frac{\Delta_4}{r_1+r_2+r_3+r_4+r_5+r_6-1}$$

where

$$\begin{aligned}
\Delta_4 \;=\;& -W_{-n+5} + (r_1-1)W_{-n+4} + (r_1+r_2-1)W_{-n+3} + (r_1+r_2+r_3-1)W_{-n+2} \\
&+(r_1+r_2+r_3+r_4-1)W_{-n+1} + (r_1+r_2+r_3+r_4+r_5-1)W_{-n} \\
&+W_5 + (1-r_1)W_4 + (1-r_1-r_2)W_3 + (1-r_1-r_2-r_3)W_2 \\
&+(1-r_1-r_2-r_3-r_4)W_1 + (1-r_1-r_2-r_3-r_4-r_5)W_0.
\end{aligned}$$

(b) *If $(r_1-r_2+r_3-r_4+r_5-r_6+1)(r_1+r_2+r_3+r_4+r_5+r_6-1) \neq 0$ then*

$$\sum_{k=1}^{n} W_{-2k} = \frac{\Delta_5}{(r_1-r_2+r_3-r_4+r_5-r_6+1)(r_1+r_2+r_3+r_4+r_5+r_6-1)}$$

where

$$\begin{aligned}
\Delta_5 \;=\;& (r_2+r_4+r_6-1)W_{-2n+4} - (r_3+r_5+(r_2+r_4+r_6)r_1)W_{-2n+3} \\
&+(r_4+r_6-(r_4+r_6)r_2+(r_3+r_5+r_1)r_1-(r_2-1)^2)W_{-2n+2} \\
&-((r_1+r_3)r_4+(r_1+r_3)r_6+(1-r_2)r_5)W_{-2n+1} \\
&+(r_6+2(r_2+r_4)+(r_1+r_3)r_5-(r_2+r_4)r_6+(r_1+r_3)^2-(r_2+r_4)^2-1)W_{-2n} \\
&-r_6(r_1+r_3+r_5)W_{-2n-1} + (r_1+r_3+r_5)W_5 - (r_2+r_4+r_6+(r_3+r_5+r_1)r_1-1)W_4 \\
&+((r_4+r_6)r_1+(1-r_2)(r_3+r_5))W_3 + (-r_4-r_6-(r_1+r_3)r_5+(r_4+r_6)r_2-(r_1+r_3)^2+(r_2-1)^2)W_2 \\
&+(r_5+(r_1+r_3)r_6-(r_2+r_4)r_5)W_1 \\
&+(-r_6-2(r_2+r_4)-2(r_1+r_3)r_5+(r_2+r_4)r_6-(r_1+r_3)^2+(r_2+r_4)^2-r_5^2+1)W_0.
\end{aligned}$$

(c)

$$\sum_{k=1}^{n} W_{-2k+1} = \frac{\Delta_6}{(r_1-r_2+r_3-r_4+r_5-r_6+1)(r_1+r_2+r_3+r_4+r_5+r_6-1)}$$

where

$$\begin{aligned}
\Delta_6 \;=\;& -(r_1+r_3+r_5)W_{-2n+4} + (r_2+r_4+r_6+(r_3+r_5+r_1)r_1-1)W_{-2n+3} \\
&+((r_2-1)(r_3+r_5)-(r_4+r_6)r_1)W_{-2n+2} \\
&+(r_4+r_6+(r_1+r_3)r_5-(r_4+r_6)r_2+(r_1+r_3)^2-(r_2-1)^2)W_{-2n+1} \\
&+(-r_5-(r_1+r_3)r_6+(r_2+r_4)r_5)W_{-2n} + r_6(r_2+r_4+r_6-1)W_{-2n-1} - (r_2+r_4+r_6-1)W_5 \\
&+(r_3+r_5+(r_2+r_4+r_6)r_1)W_4 + (-r_4-r_6-(r_1+r_3+r_5)r_1+(r_4+r_6)r_2+(r_2-1)^2)W_3 \\
&+((r_4+r_6)r_1+(r_4+r_6)r_3+(1-r_2)r_5)W_2 \\
&+(-r_6-2(r_2+r_4)-(r_1+r_3)r_5+(r_2+r_4)r_6-(r_1+r_3)^2+(r_2+r_4)^2+1)W_1 + r_6(r_1+r_3+r_5)W_0.
\end{aligned}$$

Proof. The proof is given in Soykan [13].

The following proposition presents some formulas of generalized 6-primes numbers with negative subscripts.

Proposition 5.2. *If* $r_1 = 2, r_2 = 3, r_3 = 5, r_4 = 7, r_5 = 11, r_6 = 13$ *then for* $n \geq 1,$ *we have the following formulas:*

(a) $\sum_{k=1}^{n} V_{-k} = \frac{1}{40}(-V_{-n+5} + V_{-n+4} + 4V_{-n+3} + 9V_{-n+2} + 16V_{-n+1} + 27V_{-n} + V_5 - V_4 - 4V_3 - 9V_2 - 16V_1 - 27V_0).$

(b) $\sum_{k=1}^{n} G_{-2k} = \frac{1}{80}(-11V_{-2n+4} + 31V_{-2n+3} + 4V_{-2n+2} + 59V_{-2n+1} + 36V_{-2n} + 117V_{-2n-1} - 9V_5 + 29V_4 - 4V_3 + 41V_2 + 4V_1 + 63V_0).$

(c) $\sum_{k=1}^{n} G_{-2k+1} = \frac{1}{80}(9V_{-2n+4} - 29V_{-2n+3} + 4V_{-2n+2} - 41V_{-2n+1} - 4V_{-2n} - 143V_{-2n-1} + 11V_5 - 31V_4 - 4V_3 - 59V_2 - 36V_1 - 117V_0).$

Proof. Take $r_1 = 2, r_2 = 3, r_3 = 5, r_4 = 7, r_5 = 11$ in Theorem 5.2.

From the above proposition, we have the following corollary which gives sum formulas of 6-primes numbers (take $G_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 0, G_3 = 0, G_4 = 1, G_5 = 2$).

Corollary 5.4. For $n \geq 1,$ 6-primes numbers have the following properties.

(a) $\sum_{k=1}^{n} G_{-k} = \frac{1}{40}(-G_{-n+5} + G_{-n+4} + 4G_{-n+3} + 9G_{-n+2} + 16G_{-n+1} + 27G_{-n} + 1).$

(b) $\sum_{k=1}^{n} G_{-2k} = \frac{1}{80}(-11G_{-2n+4} + 31G_{-2n+3} + 4G_{-2n+2} + 59G_{-2n+1} + 36G_{-2n} + 117G_{-2n-1} + 11).$

(c) $\sum_{k=1}^{n} G_{-2k+1} = \frac{1}{80}(9G_{-2n+4} - 29G_{-2n+3} + 4G_{-2n+2} - 41G_{-2n+1} - 4G_{-2n} - 143G_{-2n-1} - 9).$

Taking $G_n = H_n$ with $H_0 = 6,\ H_1 = 2, H_2 = 10, H_3 = 41, H_4 = 150, H_5 = 542$ in the last proposition, we have the following corollary which presents sum formulas of 6-primes -Lucas numbers.

Corollary 5.5. For $n \geq 1,$ 6-primes -Lucas numbers have the following properties.

(a) $\sum_{k=1}^{n} H_{-k} = \frac{1}{40}(-H_{-n+5} + H_{-n+4} + 4H_{-n+3} + 9H_{-n+2} + 16H_{-n+1} + 27H_{-n} - 56).$

(b) $\sum_{k=1}^{n} H_{-2k} = \frac{1}{80}(-11H_{-2n+4} + 31H_{-2n+3} + 4H_{-2n+2} + 59H_{-2n+1} + 36H_{-2n} + 117H_{-2n-1} + 104).$

(c) $\sum_{k=1}^{n} H_{-2k+1} = \frac{1}{80}(9H_{-2n+4} - 29H_{-2n+3} + 4H_{-2n+2} - 41H_{-2n+1} - 4H_{-2n} - 143H_{-2n-1} - 216).$

From the above proposition, we have the following corollary which gives sum formulas of modified 6-primes numbers (take $H_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 0, E_3 = 0, E_4 = 1, E_5 = 1$).

Corollary 5.6. For $n \geq 1,$ modified 6-primes numbers have the following properties.

(a) $\sum_{k=1}^{n} E_{-k} = \frac{1}{40}(-E_{-n+5} + E_{-n+4} + 4E_{-n+3} + 9E_{-n+2} + 16E_{-n+1} + 27E_{-n}).$

(b) $\sum_{k=1}^{n} E_{-2k} = \frac{1}{80}(-11E_{-2n+4} + 31E_{-2n+3} + 4E_{-2n+2} + 59E_{-2n+1} + 36E_{-2n} + 117E_{-2n-1} + 20).$

(c) $\sum_{k=1}^{n} E_{-2k+1} = \frac{1}{80}(9E_{-2n+4} - 29E_{-2n+3} + 4E_{-2n+2} - 41E_{-2n+1} - 4E_{-2n} - 143E_{-2n-1} - 20).$

6 MATRICES RELATED WITH GENERALIZED 6-PRIMES NUMBERS

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+5} \\ W_{n+4} \\ W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_5 \\ W_4 \\ W_3 \\ W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{6.1}$$

For matrix formulation (6.1), see [14]. In fact, Kalman give the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \\ W_{n+3} \\ W_{n+4} \\ W_{n+5} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ k_1 & k_2 & k_3 & k_4 & k_5 & k_6 \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \end{pmatrix}.$$

We define the square matrix A of order 6 as:

$$A = \begin{pmatrix} 2 & 3 & 5 & 7 & 11 & 13 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = -13$. From (1.4) we have

$$\begin{pmatrix} V_{n+5} \\ V_{n+4} \\ V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 & 7 & 11 & 13 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_{n+4} \\ V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \\ V_{n-1} \end{pmatrix}. \tag{6.2}$$

and from (6.1) (or using (6.2) and induction) we have

$$\begin{pmatrix} V_{n+5} \\ V_{n+4} \\ V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 & 7 & 11 & 13 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_5 \\ V_4 \\ V_3 \\ V_2 \\ V_1 \\ V_0 \end{pmatrix}.$$

If we take $V_n = G_n$ in (6.2) we have

$$\begin{pmatrix} G_{n+5} \\ G_{n+4} \\ G_{n+3} \\ G_{n+2} \\ G_{n+1} \\ G_n \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 & 7 & 11 & 13 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} G_{n+4} \\ G_{n+3} \\ G_{n+2} \\ G_{n+1} \\ G_n \\ G_{n-1} \end{pmatrix}. \tag{6.3}$$

We also define

$$B_n = \begin{pmatrix} G_{n+4} & \sum\limits_{k=0}^{4} r_{k+2}G_{n+3-k} & \sum\limits_{k=0}^{3} r_{k+3}G_{n+3-k} & \sum\limits_{k=0}^{2} r_{k+4}G_{n+3-k} & \sum\limits_{k=0}^{1} r_{k+5}G_{n+3-k} & r_6G_{n+3} \\ G_{n+3} & \sum\limits_{k=1}^{5} r_{k+1}G_{n+3-k} & \sum\limits_{k=1}^{4} r_{k+2}G_{n+3-k} & \sum\limits_{k=1}^{3} r_{k+3}G_{n+3-k} & \sum\limits_{k=1}^{2} r_{k+4}G_{n+3-k} & r_6G_{n+2} \\ G_{n+2} & \sum\limits_{k=2}^{6} r_kG_{n+3-k} & \sum\limits_{k=2}^{5} r_{k+1}G_{n+3-k} & \sum\limits_{k=2}^{4} r_{k+2}G_{n+3-k} & \sum\limits_{k=2}^{3} r_{k+3}G_{n+3-k} & r_6G_{n+1} \\ G_{n+1} & \sum\limits_{k=3}^{7} r_{k-1}G_{n+3-k} & \sum\limits_{k=3}^{6} r_kG_{n+3-k} & \sum\limits_{k=3}^{5} r_{k+1}G_{n+3-k} & \sum\limits_{k=3}^{4} r_{k+2}G_{n+3-k} & r_6G_n \\ G_n & \sum\limits_{k=4}^{8} r_{k-2}G_{n+3-k} & \sum\limits_{k=4}^{7} r_{k-1}G_{n+3-k} & \sum\limits_{k=4}^{6} r_kG_{n+3-k} & \sum\limits_{k=4}^{5} r_{k+1}G_{n+3-k} & r_6G_{n-1} \\ G_{n-1} & \sum\limits_{k=5}^{9} r_{k-3}G_{n+3-k} & \sum\limits_{k=5}^{8} r_{k-2}G_{n+3-k} & \sum\limits_{k=5}^{7} r_{k-1}G_{n+3-k} & \sum\limits_{k=5}^{6} r_kG_{n+3-k} & r_6G_{n-2} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_{n+4} & \sum\limits_{k=0}^{4} r_{k+2}V_{n+3-k} & \sum\limits_{k=0}^{3} r_{k+3}V_{n+3-k} & \sum\limits_{k=0}^{2} r_{k+4}V_{n+3-k} & \sum\limits_{k=0}^{1} r_{k+5}V_{n+3-k} & r_6V_{n+3} \\ V_{n+3} & \sum\limits_{k=1}^{5} r_{k+1}V_{n+3-k} & \sum\limits_{k=1}^{4} r_{k+2}V_{n+3-k} & \sum\limits_{k=1}^{3} r_{k+3}V_{n+3-k} & \sum\limits_{k=1}^{2} r_{k+4}V_{n+3-k} & r_6V_{n+2} \\ V_{n+2} & \sum\limits_{k=2}^{6} r_{k}V_{n+3-k} & \sum\limits_{k=2}^{5} r_{k+1}V_{n+3-k} & \sum\limits_{k=2}^{4} r_{k+2}V_{n+3-k} & \sum\limits_{k=2}^{3} r_{k+3}V_{n+3-k} & r_6V_{n+1} \\ V_{n+1} & \sum\limits_{k=3}^{7} r_{k-1}V_{n+3-k} & \sum\limits_{k=3}^{6} r_{k}V_{n+3-k} & \sum\limits_{k=3}^{5} r_{k+1}V_{n+3-k} & \sum\limits_{k=3}^{4} r_{k+2}V_{n+3-k} & r_6V_{n} \\ V_{n} & \sum\limits_{k=4}^{8} r_{k-2}V_{n+3-k} & \sum\limits_{k=4}^{7} r_{k-1}V_{n+3-k} & \sum\limits_{k=4}^{6} r_{k}V_{n+3-k} & \sum\limits_{k=4}^{5} r_{k+1}V_{n+3-k} & r_6V_{n-1} \\ V_{n-1} & \sum\limits_{k=5}^{9} r_{k-3}V_{n+3-k} & \sum\limits_{k=5}^{8} r_{k-2}V_{n+3-k} & \sum\limits_{k=5}^{7} r_{k-1}V_{n+3-k} & \sum\limits_{k=5}^{6} r_{k}V_{n+3-k} & r_6V_{n-2} \end{pmatrix}$$

where

$$r_1 = 2, r_2 = 3, r_3 = 5, r_4 = 7, r_5 = 11, r_6 = 13.$$

Theorem 6.1. *For all integer* $m, n \geq 0,$ *we have*

(a) $B_n = A^n.$

(b) $C_1A^n = A^nC_1.$

(c) $C_{n+m} = C_nB_m = B_mC_n.$

Proof.

(a) By expanding the vectors on the both sides of (6.3) to 6-colums and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1}B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

(b) Using (a) and definition of C_1, (b) follows.

(c) We have $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^mC_1 = A^{n-1}C_1A^m = C_nB_m$$

and similarly

$$C_{n+m} = B_mC_n. \square$$

Some properties of matrix A^n can be given as

$$A^n = 2A^{n-1} + 3A^{n-2} + 5A^{n-3} + 7A^{n-4} + 11A^{n-5} + 13A^{n-6}$$

and

$$A^{n+m} = A^nA^m = A^mA^n$$

and

$$\det(A^n) = (-13)^n$$

for all integers m and n.

Theorem 6.2. *For $m, n \geq 0,$ we have*

$$\begin{aligned} V_{n+m} &= V_n G_{m+4} + \sum_{i=1}^{6-1} V_{n-i} \left(\sum_{j=1}^{6-i} r_{j+i} G_{m+4-j} \right) \qquad (6.4) \\ &= V_n G_{m+4} + V_{n-1}(3G_{m+3} + 5G_{m+2} + 7G_{m+1} + 11G_m + 13G_{m-1}) \\ &\quad +V_{n-2}(5G_{m+3} + 7G_{m+2} + 11G_{m+1} + 13G_m) + V_{n-3}(7G_{m+3} + 11G_{m+2} + 13G_{m+1}) \\ &\quad +V_{n-4}(11G_{m+3} + 13G_{m+2}) + 13V_{n-5}G_{m+3}. \end{aligned}$$

Proof. From the equation $C_{n+m} = C_n B_m = B_m C_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and $C_n B_m$. This completes the proof. □

Remark 6.1. By induction, it can be proved that for all integers $m, n \leq 0$, (6.4) holds. So for all integers m, n, (6.4) is true.

Corollary 6.1. For all integers m, n, we have

$$G_{n+m} = G_n G_{m+4} + \sum_{i=1}^{6-1} G_{n-i} \left(\sum_{j=1}^{6-i} r_{j+i} G_{m+4-j} \right), \qquad (6.5)$$

$$H_{n+m} = H_n G_{m+4} + \sum_{i=1}^{6-1} H_{n-i} \left(\sum_{j=1}^{6-i} r_{j+i} G_{m+4-j} \right), \qquad (6.6)$$

$$E_{n+m} = E_n G_{m+4} + \sum_{i=1}^{6-1} E_{n-i} \left(\sum_{j=1}^{6-i} r_{j+i} G_{m+4-j} \right). \qquad (6.7)$$

7 CONCLUSIONS

In the literature, there have been so many studies of the sequences of numbers and the sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. We introduce the generalized 6-primes sequence (and it's three special cases, namely, 6-primes, Lucas 6-primes and modified 6-primes sequences) and we present Binet's formulas, generating functions, Simson formulas, the summation formulas, some identities and matrices for these sequences.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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