Asian Research Journal of Mathematics

Volume 19, Issue 11, Page 61-68, 2023; Article no.ARJOM.107515
ISSN: 2456-477X

A Numerical Approximation of the Stochastic Ito-Volterra Integral Equation

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2023/v19i11753

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/107515

Original Research Article

Received: 12/08/2023
Accepted: 17/10/2023
Published: 25/10/2023

Abstract

A stochastic differential equation (SDE) is a differential equation in which one or more of the terms and the solution are stochastic processes. Numerous studies have employed orthogonal polynomials, however most of them focus on deterministic rather than stochastic systems. This is the reason why in this study, we looked into a numerical solution for the stochastic Ito-Volterra integral equation using the explicit finite difference scheme and Bernstein polynomials as trial functions. The equidistant collocation procedure was used to calculate the unknown constant parameters in between and reach the desired approximation. The method was evaluated and contrasted with the Block Pulse method for approximate answers based on the aforementioned method, which were obtained and compared with others in the literature.

Keywords: *Process; Ito-Volterra integral equation; Stochastic Ito-Volterra integral equation; explicit finite difference scheme; Bernstein polynomials.*

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1 Introduction

A stochastic differential equation (SDE) is a differential equation that has one or more stochastic processes as terms, with the solution being another stochastic process. SDEs typically include a variable that is calculated as the Wiener process or Brownian motion derivative and represents random white noise. However, Bachelier is credited with one of the early works on Brownian motion in his thesis titled "Theory of Speculation" [1]. This work was followed upon by Langevin. Later Itô and Stratonovich put SDEs on more solid mathematical footing. Itô in 1944 [2], laid the foundation of a stochastic calculus known today as the Itô calculus. This represents the stochastic generalization of the classical differential calculus, which models various phenomena in continuous time such as the dynamics of stock prices, physical systems or the motions of a microscopic particle subject to random fluctuations. The corresponding stochastic differential equations (SDEs) generalize the ordinary deterministic differential equations (ODEs) [3].

In general, 1-dimensional Ito stochastic differential equation has the form [4]

$$dX_t = \alpha(X_t)dt + \beta(X_t)dW_t, \qquad t \geq 0,$$

where $\alpha(X_t)$ is called the driff coefficient (which varies slowly), and $\beta(X_t)$ is the diffusion coefficient (a rapidly varying component). W_t is a Wiener process $W = \{W_t, t \geq 0\}$ that defines the randomness of the physical system, and it is often called the white noise. The subscript t in the white noise represents time-dependence.

The origin of Ito stochastic integral was an outcome of intense investigation of properties/ conditions under which the localized properties of a Markov process may be applied to examine this process. By localized properties, we mean the drift and diffusion coefficients of an Ito process. This idea had been used by Kloeden and Platen [5,6] to implement his derivatives of differential equations (particularly, the stochastic differential equation) governing the characterization of the transition properties of an Ito process.

An ordinary differential equation is defined as a degenerate form of a stochastic differential equation, which is defined in a non chaotic dynamical system.

Thus,

$$\frac{du}{dt} = \alpha(t, x) \tag{1}$$

Rewriting (1) in its symbolic differential form, we have

$$du = \alpha(t, x)dt \tag{2}$$

Now integrating (2) in the interval $t \in [t_0, t]$, we have

$$u(t) = u_0 + \int_{t_0}^{t} \alpha\big(s, u(s)\big)ds \tag{3}$$

where $u(t) = u(t; u_0, t_0]$ is a function satisfying the initial condition $u(t_0) = u_0$.

During the first decade, Einstein and others such as Largevin offered an explanation of Brownian motion to the formulation of classical dynamics in terms of stochastic differential equation. Thus, the resulting stochastic equation can be written as

$$du_t = \alpha(t, u_t)dt + \beta(t, u_t)\xi t \tag{4}$$

where $\alpha(t, u_t)$ is a deterministic drift coefficient been perturbed by a noise term, $\beta(t, u_t)\xi t$, where ξt defines a standard Wiener process for each variables and $\beta(t, u_t)$ is a space – time (dependent) intensity term. Equation (4) can be reformulated as an integral equation of the form

$$\boldsymbol{u_t(\omega) = u_{t_0}(\omega) + \int_{t_0}^{t} \alpha(s, u_s(\omega)ds + \int_{t_0}^{t} \beta(s, u_s(\omega)\xi s(\omega)ds}\ , \tag{5}$$

defining each trajectory or sample path.

Now a special case for (5) with $\alpha = 0 \; and \; \beta = 1$, we see that ξ_t is a bounded real Brownian motion, thus (5) can be alternatively be written as

$$u_t(\omega) = u_{t_0}(\omega) + \int_{t_0}^{t} \alpha(s, u_s(\omega) ds + \int_{t_0}^{t} \beta(s, u_s(\omega)(\omega) dW_s(\omega), \tag{6}$$

where W_t is a white noise process. W_t is not completely dependent on t since it is nowhere differentiable. This implies that the second integral in (6) can be Lebesgue or Riemann integral. In fact, the second integral in (6) cannot be even interpreted as Riemann – Stieltjes integral since the W_t is unbounded at any bounded time interval for each trajectory or sample path. For $\beta(t, u) = \beta(a \; constant)$, we expect the second integral in (6) to be defined and equal to $\beta\{W_t(\omega) - W_{t_o}(\omega)\}$.This is the origin for definition of an Ito stochastic integral.

There is a wide applicability of stochastic integral equations (SIE) in oceanography, physical sciences, engineering, etc, which are noise driven such as the white noise [7] and [8]. Most SIEs do not have an exact solver due to the presence of the noise, thus the role of numerical methods for computing an accurate and reliable numerical approximation have become imperative [9].

In this research, we shall consider the Stochastic Ito-Volterra integral equation of the form [8]

$$U(t) = g(t) + \int_0^t y(s,t)U(s)ds + \sum_{j=1}^{n} \int_0^t B_j(s,t)U(s)dB_j(s)ds, \tag{7}$$

where $t \in \Omega = (0, T], U, g, y$ and $B_j, \; j = 1(2)n, s, t \in \Omega$ are stochastic processes enclosed on the probability space $(\Omega, \mathrm{F}, \mathrm{P})$ and U is unknown. Also, $\int_0^t B_j(s,t)U(s)dB_j(s)ds, \; j = 1(2)n,$ is the Itô integral and $B_j(t), j = 1(2)n$ are the Brownian motion processes.

In recent years, there are few different numerical techniques for solving most stochastic Ito-Volterra integral equations (SIVIE), such as, the wash series method, polynomial method, block pulse method, hat function method, triangular method and orthogonal basis method. For instance, Maleknejadet al.[4] solved the m-multidimensional SIVIE using the block pulse method via the operational matrix scheme. Also, the convergent of the method was shown to be $o(h)o(h)$. Bentol et al. [10] solved the solution of linear stochastic SIVIE driven by fractional Brownian motion with Hurst parameter $H \in (0,1)$ via the operational matrix block pulse method with hat functions as basis. Khodabin et. al. [11] applied the triangular functions to numerical solution of stochastic Volterra integral equations.

However, there is a major gap in literature. The use of orthogonal basis via a discretization scheme such as the finite difference method for seeking the solution of stochastic Ito-Volterra integral equations has not been exploded. Thus, this research will adopt the Bernstein polynomials as orthogonal basis in a finite difference discretization approach to solve the Stochastic Ito-Volterra integral equations.

2 Materials and Methods

2.1 Preliminaries

Definition 2.1 (*Bernstein Polynomials).* These are polynomials defined as [12-20]

$$B_{i,n} = \frac{n!}{i(n-i)!}(1-t)^{n-i}t^i, i = 0(1)n. \tag{8}$$

Some basic properties of the Bernstein polynomials include;

i. Positive definiteness: $B_{i,n}(t) \geq 0, \; t \in [0,1], \; i = 0(1)n.$
ii. Partition of unity: $B_{i,n}(t) = \sum_{i=0}^{n} B_{i,n}(t) = 1.$

iii. Symmetry: $B_{i,n}(t) = B_{n-i,n}(1-t)$.
iv. Linear precision: $t = \sum_{i=0}^{n} \frac{i}{n} B_{i,n}(t)$.
v. Recurrence relation: $B_{i,n}(t) = (1-t)B_{i,n-1}(t) + tB_{i-1,n-1}(t)$, $B_{i,n}(t) = 0, i < 0, i > 0, B_{0,0}(t) = 1$.

Definition 2.2 (*Ito stochastic Formula)*

Let e and g be two arbitrary functions into $|\rho|^{\frac{1}{2}}$ and $g \in L_T^2$, so that e and g are well defined in L_T^W with all satisfactory properties except if $e(t,z)$ is replaced by (Farnoosh, *et al*., 2015)

$$\int_0^T |e(s,z)|ds < \infty.$$

By SDE, we mean an expression of the form

$$du_t = e(t,z)dt + g(t,z)dW_t(z),$$

which can be written as

$$u_t(z) - u_s(z) = \int_s^t e(x,z)dx + \int_s^t g(x,z)\, dW_x(z) \tag{9}$$

for $0 \le s \le t \le T$. Here, the first integral is termed Lebesgue or an ordinary Riemann integral for all $z \in \mathbf{\Omega}$, and the second is an Ito integral.

3 Finite Difference Method

The principle of finite difference methods is close to the numerical schemes used to solve ordinary differential equations. It majors in approximating the differential operator by replacing the derivatives in the equation using differential quotients. The domain is partitioned in space and in time and approximations of the solution are computed at the space or time points. The error between the numerical solution and the exact solution is determined by the error that is committed by going from a differential operator to a difference operator. This error is called the discretization error or truncation error. The term truncation error reflects the fact that a finite part of a Taylor series is used in the approximation [21].

For the sake of simplicity, we shall consider the one-dimensional case only. The main concept behind any finite difference scheme is related to the definition of the derivative of a smooth function f at a point ∈ R,

$$f'(x) = \lim_{h \to \infty} \frac{f(x+h) - f(x)}{h}$$

and to the fact that when h tends to 0 (without vanishing), the quotient on the right-hand side provides an "appropriate" approximation of the derivative. In other words, h should be sufficiently small to get a good approximation. It remains to indicate what exactly a good approximation, in what sense is. Actually, the approximation is good when the error committed in this approximation (i.e. when replacing the derivative by the differential quotient) tends towards zero when h tends to zero. If the function u is sufficiently smooth in the neighborhood of x, it is possible to quantify this error using a Taylor expansion.

3.1 Finite difference method for stochastic Ito-Volterra integral equations with Bernstein polynomials

Given the Stochastic Ito-Volterra Integral Equations (SI-VIE)

$$U(t) = g(t) + \int_0^t y(s,t)U(s)ds + \sum_{j=1}^{n} \int_0^t B_j(s,t)U(s)dB_j(s)ds, \tag{8}$$

where $t \in \Omega = (0,T], U, g, y$ and B_j, $j = 1(2)n, s, t \in \Omega$ are stochastic processes enclosed on the probability space $(\Omega, \mathrm{F}, \mathrm{P})$ and $U(s)$ is unknown. Rewriting (8) in its differential form yields,

$$U'(t) = g'(t) + [y(t,t)U(t) - y(0,t)U(0)] + \sum_{j=1}^{n}[B_j(t,t)U(t)dB_j(t) - B_j(0,t)U(0)dB_j(0)] \quad (9)$$

Now, using a forward difference at time $t_n$on (9), we get the recurrence equation:

$$\frac{U_j^{(n+1)} - U_j^{(n)}}{k} = g'(t) + \left[y(t,t)U_j(t) - y(0,t)U_j(0)\right] + \sum_{j=1}^{n}[B_j(t,t)U_j(t)dB_j(t) - B_j(0,t)U_j(0)dB_j(0)] \quad (10)$$

This is an explicit method for solving the Stochastic Ito-Volterra Integral Equations.

Let

$$U_j^{(n)}(t) = \sum_{i=0}^{n} c_i B_{i,n}, \quad (11)$$

be an approximate solution of (10), where $c_i's$ are constants, and $B_{i,n}$ uniquely defined by (8).

Thus, (10) becomes

$$\sum_{i=0}^{n+1} c_i B_{i,n+1}, = \sum_{i=0}^{n} c_i B_{i,n}, + k\big(g'(t) + \left[y(t,t)U_j(t) - y(0,t)U_j(0)\right] + \sum_{j=1}^{n}[B_j(t,t)U_j(t)dB_j(t) - B_j(0,t)U_j(0)dB_j(0)]\big), \quad (12)$$

where $U_j^{(n)}(t) \cong U_j(t)\ at\ n = 0,$ and

$$\mathrm{d}B_j = B_{\gamma_{n+1}} - B_{\gamma_n} \ \text{ for } j = 0(1)(j-1), \quad (13)$$

such that $0 = \gamma_0 < \gamma_1 < \gamma_2 < \cdots < \gamma_N = t, j \geq 0,$ over the time interval $[0,t]$ with $\delta = \frac{t}{j}$, being the equidistant step size.

Evaluating (12) for $n \geq 0$, and collocating via equidistant step size procedure we arrive at system of $(n+1)$ equation with n unknowns. Solving the resulting systems yields the unknown $c_i's$, which are substituted into (11) for the approximate solution.

4 Numerical Simulations

To illustrate the method, we consider the example of which the analytic solutions exist with the help of MAPLE 18.

Example

Consider the Stochastic Ito-Volterra Integral Equations:

$$U(t) = -\frac{1}{8} - \int_0^t \frac{1}{4} s \times U(s)ds - \int_0^t \frac{1}{40} U(s)dB_j(s), t \in [0,T], T < 1.$$

The analytic solution is as:

$$U(t) = -\frac{1}{8} e^{\left(-\frac{1}{40}B_j(t) - \frac{t^3}{8} - \frac{1}{3200}t^2\right)}$$

A comparison between the exact and approximation of the solution by methodology above is given in Table 1 for $t = 0.05, 0.1, 0.15$ and Fig. 1. for $n = 16$.

Table 1. Comparison of results between exact and computed

t	Exact	Computed	Error
0.1	-0.1249999	-0.1249999	6.2500e-08
0.2	-0.1249998	-0.1249999	9.3700e-08
0.3	-0.1249998	-0.1249999	1.2500e-07
0.4	-0.1249997	-0.1249998	1.5630e-07
0.5	-0.1249996	-0.1249998	1.8750e-07
0.6	-0.1249996	-0.1249998	2.1870e-07
0.7	-0.1249995	-0.1249998	2.5000e-07
0.8	-0.1249994	-0.1249997	2.8130e-07
0.9	-0.1249994	-0.1249997	3.1250e-07

Table 2. Maximum absolute errors

t	Present error	Block pulse method [9] error
0.05	3.037852E-07	3.8000000E-05
0.1	80.08524E-08	1.0450000E-04
0.15	9.136447E-07	9.6500000E-04

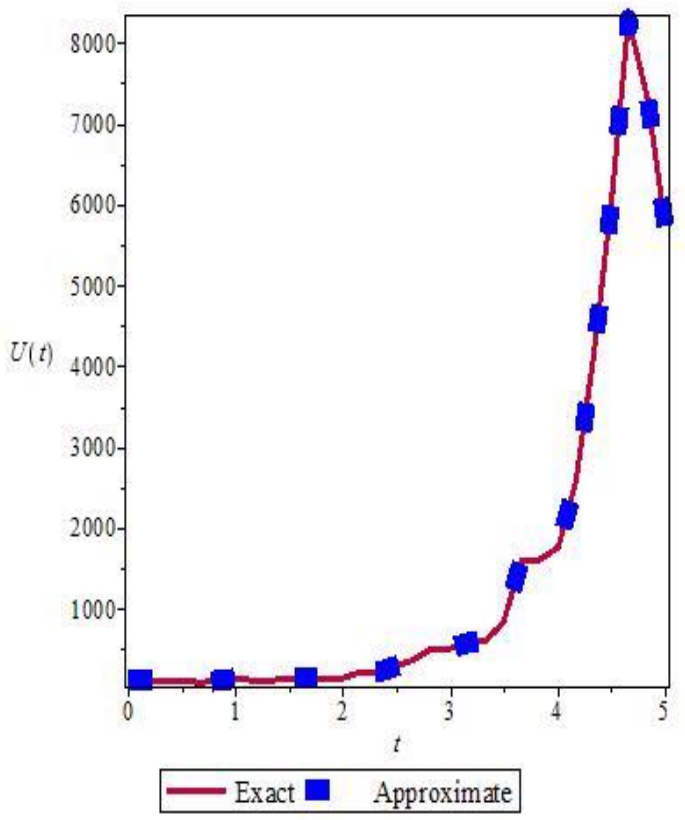

Fig. 1. Graphical view of the exact and approximate solution

5 Conclusions

We have considered in this paper the numerical approximation of the Stochastic Ito-Volterra Integral equation via the Bernstein polynomials as basic functions. Consequently, the numerical results showered in Table 1 and 2 showing the comparison of Results between Exact and Computed with the maximum absolute errors shows the accuracy of the method which performs better than the Block Pulse method in Bentol et. al. [9]

Competing Interests

Authors have declared that no competing interests exist.

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