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A New Six Dimensional Representation of the Braid Group on Three Strands and its Irreducibility and Unitarizability

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Research Article

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Abstract

We consider the braid group on three strands, B_3 and construct a complex valued representation of it with degree 6, namely, $\rho : B_3 \to GL_6(\mathbb{C})$. First, we show that this representation is irreducible and not equivalent to either Burau or Krammer's representations. Second, we prove that the representation is unitary relative to an invertible hermitian matrix.

Keywords: Krammer's representation; Artin representation; braid group; Hecke algebra
2010 Mathematics Subject Classification: Primary: 20F36

1 Introduction

Let B_n be the braid group on n strands. This group has a standard presentation

$$< \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i, \text{ if } \mid i-j \mid > 1; \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \le i \le n-2 > .$$

There is a well known representation of the braid group B_n, due to Artin, in the group $Aut(F_n)$ of automorphisms of a free group F_n generated by $x_1, \ldots, x_n$ [1].

Researchers gave a great value for representations of the braid group. Burau was the first who constructed non trivial representations of B_n of degrees n and $n-1$, known as Burau and reduced Burau representations respectively [3]. The reduced Burau representation was proved to be irreducible and not faithful for $n > 4$ [8]. Moreover, D. Krammer constructed an irreducible and faithful representation of B_n of degree $\frac{n(n-1)}{2}$ with 2 indeterminates and thus, he solved the outstanding problem of linearity of the braid group [6]. Recently, some researchers construct representations of the braid group of high degree such as the spin representation constructed by Paul Tian [11].

In this paper, we construct a complex valued irreducible representation of the braid group on 3 strands of degree 6 which is a subrepresentation of the spin representation with negative index equal to

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one [11]. What distinguishes our representation from other known ones is that it is an irreducible representation that doesn't arise from any Hecke algebra. It is also not equivalent to either Burau representation or Krammer's representation.

In section 2, we give the matrix images of the generators of the braid group on three strands under the constructed representation $\rho : B_3 \to GL_6(\mathbb{C})$. We prove that it is an irreducible representation and that it is not equivalent to either Burau representation or Krammer's representation.

In section 3, we show that $\rho : B_3 \to GL_6(\mathbb{C})$ is unitary relative to an invertible and hermitian matrix. This is analogous to previous results done concerning Burau and Krammer's representations [7] and [10].

In section 4, we make a discussion explaining the possibility of constructing many irreducible representations of the braid group on "n" strands. (Here $n \geq 3$)

2 $\rho : B_3 \to GL_6(\mathbb{C})$ is irreducible

In this section, we construct a representation of the braid group on three strands, namely $\rho : B_3 \to GL_6(\mathbb{C})$, and we prove that it is irreducible and not equivalent to either Burau or Krammer's representations. This representation is a subrepresentation of the spin representation with negative index equal to one, namely $\alpha : B_3 \to GL_9(\mathbb{C})$ [11]. We reduce α to a representation of degree 8 which can be written further as a direct sum of $\rho : B_3 \to GL_6(\mathbb{C})$ and a representation of degree 2 (which is of Burau type).

Definition 2.1. Let z be a non zero complex number such that $z^2 \neq 1$. $\rho : B_3 \to GL_6(\mathbb{C})$ is given by:

$$\rho(\sigma_1) = \begin{pmatrix} 1-z & z & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z^{-1} & 0 & 0 \\ 0 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1-z^{-1} & 1 & 0 \end{pmatrix}$$

and

$$\rho(\sigma_2) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & z^{-1} & 0 & 0 & z^{-1} \\ 1 & z-1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & z & 0 \\ 0 & 0 & 0 & 0 & -z & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

It is easy to see that $\rho : B_3 \to GL_6(\mathbb{C})$ is a representation of B_3 as ρ satisfies the braid relations, namely $\rho(\sigma_1)\rho(\sigma_2)\rho(\sigma_1) = \rho(\sigma_2)\rho(\sigma_1)\rho(\sigma_2)$.Let G be a group with some generators. A representation $G \to GL_r(\mathbb{C})$ is reducible if there exists a non zero proper subspace of $\mathbb{C}^r$ that is invariant under the action of the generators of G.

Lemma 2.1. *Let $z \in \mathbb{C}^* - \{\pm 1\}$ and S be an invariant subspace of $\rho : B_3 \to GL_6(\mathbb{C})$. Then $S = \mathbb{C}^6$ under any of the following cases:*

1. $e_i \in S$ *for* $i = 1, 2, \ldots, 5$ *or* 6
2. $e_1 + ue_2 \in S$, $u \in \mathbb{C}^*$

3. $e_5 + ue_6 \in S$, $u \in \mathbb{C}^*$

4. $e_1 + e_2 + u(e_5 + e_6) \in S$, $u \in \mathbb{C}^*$

Proof. Let S be invariant subspace of $\rho : B_3 \to GL_6(\mathbb{C})$.

1. We consider 6 cases for e_i.

   Case 1. $e_1 \in S$. Having that $\rho(\sigma_1)e_1 = (1 - z)e_1 + e_2 \in S$ implies that $e_2 \in S$. And that $\rho(\sigma_1\sigma_2\sigma_1)e_1 = ze_5 \in S$ implies that $e_5 \in S$.
   Also, $\rho(\sigma_1\sigma_2\sigma_1)e_2 = z^2e_4 - ze_5 \in S$ implies that $e_4 \in S$.
   Having $\rho(\sigma_1)e_5 = e_6 \in S$ and $\rho(\sigma_2)e_2 = (z - 1)e_3 + e_6 \in S$, we get that $S = \mathbb{C}^6$.

   Case 2. $e_2 \in S$. Having that $\rho(\sigma_1)e_2 = ze_1 \in S$ implies that $e_1 \in S$ and by the previous case, we get that $S = \mathbb{C}^6$.

   Case 3. $e_3 \in S$. We have that $\rho(\sigma_1)e_3 = ze_4 \in S$ then $e_4 \in S$ and by acting with σ_1 on e_4, we see that e_6 is also in S. Once e_6 is in S, then by acting with σ_2, we get that e_2 is in S.

   Case 4. $e_4 \in S$. Having that $\rho(\sigma_1)e_3 = ze_4 \in S$, we get that $e_3 \in S$. We then conclude by Case 3.

   Case 5. $e_5 \in S$. We have that $\rho(\sigma_1)e_5 = e_6 \in S$ and that $\rho(\sigma_2)e_6 = \frac{e_2}{z} \in S$. Thus we return to case $e_2 \in S$.

   Case 6. $e_6 \in S$. We have that $\rho(\sigma_2)e_6 = \frac{e_2}{z} \in S$. Using the case $e_2 \in S$, we get that $S = \mathbb{C}^6$.

2. Assume that $e_1 + ue_2 \in S$. We have that $\rho(\sigma_2^2)(e_1 + ue_2) - (e_1 + ue_2) \in S$.
   We get that $(z - 1)ue_1 \in S$. Having $z \neq 1$ and $u \neq 0$ assert that $e_1 \in S$ and by (1), we get that $S = \mathbb{C}^6$.

3. Assume that $e_5 + ue_6 \in S$. We have that $\rho(\sigma_1)(e_5 + ue_6) - (e_5 + ue_6) \in S$.
   We get that $(u - 1)(e_5 - e_6) \in S$. Either $u = 1$ or $e_5 - e_6 \in S$.

   - $u = 1$ (or $e_5 + e_6 \in S$).
     We have that $\rho(\sigma_1\sigma_2\sigma_1)(e_5 + e_6) = e_1 + e_3 - e_6 \in S$ and that
     $\rho(\sigma_2\sigma_1)(e_5 + e_6) = \frac{1}{z}e_2 + ze_4 - ze_5 \in S$.
     We get that $\rho(\sigma_2^2)(e_5 + e_6) + z(\frac{1}{z}e_2 + ze_4 - ze_5) + \frac{1}{z}(e_1 + e_3 - e_6) \in S$.
     This implies that $t = e_1 + ze_2 + ze_3 + z^2e_4 \in S$.
     We have that $\rho(\sigma_1^2)t - t = (z - 1)^2(e_2 - ze_1) + z(z - 1)(e_5 + e_6) \in S$.
     Since $e_5 + e_6 \in S$ and $z \neq 1$, it follows that $e_2 - ze_1 \in S$. Using (2), we get that $S = \mathbb{C}^6$.

   - $e_5 - e_6 \in S$.
     We have that $\rho(\sigma_2\sigma_1)(e_5 - e_6) = \frac{e_2}{z} - ze_4 + ze_5 \in S$ and $\rho(\sigma_1\sigma_2\sigma_1)(e_5 - e_6) = e_1 - e_3 + e_6 \in S$.
     We get that $\rho(\sigma_2^2)(e_5 - e_6) - z(\frac{e_2}{z} - ze_4 + ze_5) + \frac{1}{z}(e_1 - e_3 + e_6) \in S$.
     This implies that $q = e_1 - ze_2 - ze_3 + z^2e_4 \in S$.
     $\rho(\sigma_1^2)q - q = (z^2 - 1)(ze_1 - e_2) + z(z - 1)(e_5 - e_6) \in S$.
     Since $e_5 - e_6 \in S$ and $z^2 \neq 1$, it follows that $ze_1 - e_2 \in S$. Using (2), we get that $S = \mathbb{C}^6$.

4. Assume that $e_1 + e_2 + u(e_5 + e_6) \in S$. We have that $\rho(\sigma_1\sigma_2)(e_1 + e_2 + u(e_5 + e_6)) = r \in S$.
   Here $r = ue_1 + ue_3 + z^2e_4 - ue_6$.
   Also, $\rho(\sigma_1^2)r - r = uz(z - 1)e_1 + u(1 - z)e_2 + z(z - 1)e_5 + u(z - 1)e_6 \in S$.
   Since $z \neq 1$, it follows that $w = uze_1 - ue_2 + ze_5 + ue_6 \in S$.
   Having $z^2 \neq 1$ and $\rho(\sigma_1^2)w - w = u(z^2 - 1)(ze_1 - e_2) \in S$, we get that $ze_1 - e_2 \in S$. Using the results in (2), we get that $S = \mathbb{C}^6$. □

Theorem 2.2. *$\rho : B_3 \to GL_6(\mathbb{C})$ is irreducible.*

Proof. Suppose for contradiction that $\rho : B_3 \to GL_6(\mathbb{C})$ is reducible and that S is a non zero proper invariant subspace of $\mathbb{C}^6$ under ρ.
Let $r = ae_1 + be_2 + ce_3 + de_4 + fe_5 + ge_6$ be a non zero vector in S. We have that $\rho(\sigma_1^2)r = (a + z(z-1)a + z(1-z)b)e_1 + ((1-z)a + zb)e_2 + ce_3 + de_4 + (f + (\frac{z-1}{z})d)e_5 + ((z-1)c + g)e_6 \in S$.
Since $\rho(\sigma_1^2)r$ and r are both in S, it follows that $\rho(\sigma_1^2)r - r \in S$.
We get that $w = z(a-b)e_1 - (a-b)e_2 + \frac{d}{z}e_5 + ce_6 \in S$.
It suffices to show that if one of the following vectors lie in S then $S = \mathbb{C}^6$

1. e_i for $i = 1, \ldots, 6$.
2. $e_5 + ue_6$, $u \neq 0$.
3. $ae_1 + be_2 + ce_5 + de_6$, $(a, b, c, d) \neq (0, 0, 0, 0)$.

(1) and (2) are proved in Lemma 2.1. To prove (3), apply again $\rho(\sigma_1^2) - I_6$ but this time to w. It yields

$$(a-b)(z+1)(ze_1 - e_2) \in S.$$

If $ze_1 - e_2$ belongs to S then S is the whole space by Lemma 2.1, point (2). Else, we must have $a = b$ since z is forbidden to take the value -1. It follows that

$$\frac{d}{z}e_5 + ce_6 \in S.$$

Then points (1) and (3) in Lemma 2.1 imply again that S is the whole space $\mathbb{C}^6$, unless $c = d = 0$. But if all the coefficients in w are zero then the original vector r which lies in S is simply $r = a(e_1 + e_2) + fe_5 + ge_6$. Without loss of generality, set $a = b = 1$. Then $r = e_1 + e_2 + fe_5 + ge_6 \in S$.
We have that $\rho(\sigma_1)r - r \in S$. Then $(g-f)(e_5 - e_6) \in S$. Either $g = f$ or $(e_5 - e_6) \in S$.
If $g = f$ then we return to (4) in Lemma 2.1 and if $e_5 - e_6 \in S$ then we return to (3) in Lemma 2.1. ☐

Definition 2.2. The Hecke algebra $H_n(q)$ is the complex algebra defined by the presentation

$$< s_1, \ldots, s_n \mid s_i s_j = s_j s_i, \mid i - j \mid > 1, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, (s_i)^2 = (1-q)s_i + q > .$$

Here, q is any nonzero complex number.

It is clear that a hecke algebra representation has at most two distinct eigenvalues.

Lemma 2.3. *$\rho : B_3 \to GL_6(\mathbb{C})$ doesn't arise from any Hecke algebra.*

Proof. Since $\rho : B_3 \to GL_6(\mathbb{C})$ has three distinct eigenvalues 1, -1 and $-z$ ($z^2 \neq 1$), it follows that ρ doesn't arise from any Hecke algebra. ☐

Corollary 2.4. *The Burau representation and $\rho : B_3 \to GL_6(\mathbb{C})$ are not equivalent.*

Proof. Using Lemma 2.3 and that the Burau representation arises from the Hecke algebra $H_n(q)$, we get that the two representations can not be equivalent. ☐

By personal communication, D. Wales and C. Levaillant gave an argument that shows that our representation and Krammer's representation of B_3 are not equivalent. Hence, we have the next Lemma.

Lemma 2.5. *The restricted Krammer's representation to B_3 of dimension 6 and our representation ρ are not equivalent.*

Proof. By restricting the Lawrence- Krammer's representation of B_4 to B_3, we get that $K : B_3 \to GL_6(\mathbb{C})$ is reducible having an invariant subspace $< e_1, e_2, e_3 >$ [7, p.21]. But our representation ρ is irreducible by Theorem 2.2 which follows that the two representations cannot be equivalent. ☐

3 $\rho : B_3 \to GL_6(\mathbb{C})$ is unitary

In this section, we find a unique matrix M in which $\rho : B_3 \to GL_6(\mathbb{C})$ is unitary relative to M. Here z is a parameter on the unit circle and not equal to 1 or -1. Since ρ is irreducible (Theorem 2.2), it would then follow that the matrix obtained is unique up to scalar multiplication.

Notation 3.1. *Let* $(*) : M_m(\mathbb{C}[t^{\pm 1}])$ *be an involution defined as follows:*

$$(f_{ij}(t))^* = f_{ji}(t^{-1})\ ,\ f_{ij}(t)) \in \mathbb{C}[t^{\pm 1}].$$

Here, t *is a complex number on the unit circle.*

Definition 3.1. Let H and U be elements of $Gl_6(\mathbb{C})$. U is called unitary relative to H if $UHU^* = H$.

Theorem 3.2. *Let* z *be a complex number on the unit circle not equal to* 1 *nor* -1. $\rho : B_3 \to GL_6(\mathbb{C})$ *is unitary relative to a unique invertible hermitian matrix.*

Proof. Let $M = \begin{pmatrix} 0 & 1+z & 0 & 0 & 0 & 0 \\ 1+z^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+z^{-1} \\ 0 & 0 & 0 & 0 & 1+z & 0 \\ 0 & 0 & 0 & 1+z^{-1} & \frac{-(1+z)^2}{z} & 0 \\ 0 & 0 & 1+z & 0 & 0 & \frac{-(1+z)^2}{z} \end{pmatrix}$.

It is easy to see that M is invertible and hermitian as Det(M)= $\frac{-(1+z)^6}{z^3} \neq 0$ and $M^* = M$. Simple computations show that $\rho(\sigma_1)M\rho(\sigma_1)^* = \rho(\sigma_2)M\rho(\sigma_2)^* = M$. ☐

4 Remarks and Discussions

Generalizing our work in this paper, we may start with any value of n ($n \geq 3$) and consider the corresponding spin representation of B_n of which the degree is n^2. We then reduce it further to an irreducible subrepresentation of a lower degree. Hence, for various values of integers n, we can get many irreducible representations of B_n. Our representation "ρ" is one of those irreducible representations thus obtained in the case $n = 3$.

Another way of finding irreducible representations of B_6, the braid group on 6 strands, is to consider the classical surjection $B_6 \to Sp_4(\mathbb{Z})$, where $Sp_4(\mathbb{Z})$ is the symplectic group consisting of all 4×4 integer matrices $\mu = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and d are 2×2 matrices. These matrices should satisfy the following relation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} 0 & -I_2 \\ -I_2 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -I_2 \\ -I_2 & 0 \end{pmatrix}$$

Here T is the transpose [12, p.8].

Direct calculations show that our representation "ρ" of dimension 6 does not arise from the surjection map restricted to B_3, the braid group on 3 strands, because the 6×6 matrices do not satisfy the relation in $Sp_4(\mathbb{Z})$, mentioned above.

Competing Interests

The authors declare that no competing interests exist.

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